Option pricing; stochastic volatility, Lévy models and Asian options

Opties prijzen; stochastische volatiliteit, Lévy modellen and Aziatische opties

Proefschrift voorgelegd tot het behalen van de graad van doctor in de Wetenschappen aan de Universiteit Antwerpen te verdedigen door Damiaan Lemmens.

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0.1 Het prijzen van opties; stochastische volatiliteit, Lévy modellen en Aziatische opties
(Nederlandstalige samenvatting)

In deze thesis wordt het prijzen van opties bestudeerd. Een optie geeft het recht om
gedurende een zekere looptijd een onderliggende, het asset genoemd, te kopen of te
verkopen tegen de uitoefenprijs. Optiecontracten kunnen verder gespecificeerd wor-
den met allerlei randvoorwaarden. Sommige opties kunnen enkel uitgeoefend worden
op het einde van de looptijd (Europese opties), andere opties worden waardeloos als
de prijs van het asset tijdens de looptijd een bepaalde grens overschrijdt (barrière-
opties), etc. Als er geen verdere voorwaarden aan het contract verbonden zijn, noemt
men het een vanille-optie. Wat het asset is wordt in dit werk meestal verzwegen,
men kan zich inbeelden dat het om staal, een wisselkoers of de brokaatkarper gaat.
Er zal aangenomen worden dat de prijs van het asset een zeker stochastisch proces
volgt, dit zal ons in staat stellen de prijs van de optie te berekenen.

Het Heston-model is een populaire manier om de dynamica van het asset te
beschrijven. In dit model bestaat de tijdsevolutie van het asset uit twee delen. Enerzijds wordt aangenomen dat de toename van de prijs van het asset evenredig is
met de prijs zelf, de evenredigheidsfactor noemt men de drift. Anderzijds worden er
fluctuaties toegevoegd waarvan de grootte eveneens evenredig is met de prijs, deze
evenredigheidsfactor noemt men de volatiliteit. Deze volatiliteit is in het Heston
model zelf een stochastische variabele. Er wordt aangenomen dat de volatiliteit
op een niet-deterministische manier rond een bepaalde waarde schommelt. In het
standaard Heston-model wordt aangenomen dat de rente constant is, wat niet het
geval is. Een mogelijke verbetering is het invoeren van stochastische rente, zodat
dee evenals de volatiliteit op een niet-deterministische manier rond een bepaalde
waarde schommelt. De eerste bijdrage van deze thesis is het opstellen van een
formule voor een Europese vanille-optie in het Heston-model met stochastische rente.

De tweede bijdrage van deze thesis is het afleiden van een formule voor geo-
metrische Aziatische opties in het Black-Scholes-model. Een Aziatische optie is
afhankelijk van de gemiddelde waarde van de prijs van het asset gedurende (een
deel van) de looptijd van de optie. De gemiddelde waarde kan aritmetisch of geo-
metrisch gedefinieerd zijn. Het Black-Scholes-model is het meest gekende model
om prijsevolutie van het asset te beschrijven, het Heston-model is hier een uitbreid-
ing van. Als men het Heston-model beperkt door de volatiliteit constant te houden,
bekomt men het Black-Scholes-model.

Een belangrijke klasse van modellen zijn de sprong-diffusie-modellen. Hierbij
heeft de prijs van het asset de mogelijkheid om grote, plotselop- of neerwaartse
sprongen te maken. Als derde onderwerp van deze thesis combineren we deze mod-
ellen met stochastische volatiliteitsmodellen. We construeren een methode om pri-
jsformules voor stochastische volatiliteitsmodellen uit te breiden tot resultaten voor
deze modellen uitgerust met sprongen in de assetprijs. De volatiliteit beschrijven
we ditmaal met het exponentieel Vašiček-model. Voor de sprongen in de assetprijs
kijken we naar het Kou- en het Merton- model.

Eerder bespraken we reeds geometrische Aziatische opties. In de praktijk zijn
aritmetische Aziatische opties relevanter. Ook wordt er meestal met discrete gemiddelde

delen gewerkt, men kan zich in de praktijk immers slechts beroepen op een eindig

aantal prijzen voor het uitrekenen van het gemiddelde. Daarom willen we ook voor
discrete aritmetische Aziatische opties prijsformules afleiden. Ditmaal concentreren
we ons op Lévy-modellen, waar de sprong-diffusie modellen bijhoren. Het blijkt niet
haalbaar om exacte prijsformules te bepalen voor de vooropgestelde modellen en het
vooropgestelde optietype, wel om onder- en bovengrenzen te bepalen. We stellen for-
mules op voor de volgende Lévy-modellen: het Kou-model, het Merton-model, het
normaal invers Gaussisch model, het variantie-gamma model en het CGMY-model.
Dit is het vierde onderwerp van deze thesis.

Als vijfde en laatste onderwerp behandelt deze thesis de geïmprimeerde kansverdel-
ing. Dit is in zekere mate het omgekeerde probleem van wat eerder besproken werd.
Ditmaal wordt de kansverdeling van de assetprijs bepaald, bij gegeven Europese
vanille-optieprijzen. Hiervoor is het theoretisch noodzakelijk voor elke uitofeping
over een optieprijs te beschikken. In de praktijk beschikt men echter maar voor
een beperkt aantal uitofepingen over een optieprijs en bevatten de data ruis.
We vinden dat de in de literatuur beschreven methodes niet voldoende getest zijn.
Daarom testen we twee populaire methodes om de geïmprimeerde kansverdeling af
te leiden. Bovendien creëren we hiervoor zelf een methode en testen deze. Om deze
methodes te testen kiezen we drie kansverdelingen: de lognormale, één gebaseerd
op het Heston-model en één gebaseerd op het CGMY-model. Vervolgens berekenen
we de optieprijzen overeenkomend met deze kansverdelingen. Bij deze optieprijzen
tellen we random ruis bij. Tenslotte bepalen we uitgaande van deze optieprijzen de
oorspronkelijke kansverdelingen met behulp van de te testen methodes. Een goede
methode zou de oorspronkelijke kansverdeling moeten terug geven.

0.2 Dankwoord

Met plezier dank ik iedereen die tot deze thesis heeft bijgedragen.
Chapter 1

Introduction*

The concept of an option is very easy: an option is a contract which provides the owner the right to buy or sell something, an “asset”, at a certain time in the future for a predetermined price. The owner is not required to buy or sell, he has the right to do so. The owner of an option contract is clearly in a profitable situation. Therefore options are not free and the obvious question is: how much is an option worth? This is the topic of this thesis.

Clearly some assumptions must be made concerning the evolution of the price of that asset, also called an underlying. The asset can be anything that can be traded and the time evolution of its price will typically be as random and complex as illustrated in figure 1.1. To determine an option price it is not necessary to predict what the asset price will be at a certain future time. As figure 1.1 already suggests, it is generally accepted that the random nature of asset prices would not allow this. To determine an option price it is enough to know in which way the asset price is random. Unfortunately also this is unknown. Therefore one tries to approximate the randomness of asset prices.

The benchmark theory to price options is the work of Black, Scholes and Merton [1, 2]. In this framework the logarithm of the asset price is modeled by a Brownian motion [3]. This elementary form of random behavior is not able to catch the so-called stylized facts of the asset price dynamics. Stylized facts such as volatility† clustering and large price movements can already be observed in figure 1.1, for more information we refer to [4–8]. Despite these shortcomings the simple diffusion model of Black and Scholes (BS) is still widely used. Much of its success is due to the availability of closed-form analytical pricing formulas for many types of derivatives [9].

To increase the practical usefulness of more realistic models, among other things, fast to evaluate pricing formulas for derivative prices within these models are needed. This is what this thesis will try to contribute to: the pricing of options in more complex models than the Black, Scholes and Merton model.

A popular class of more realistic models are the stochastic volatility models. In the Black, Scholes and Merton model the logarithm of the asset price is governed

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*This work is supported financially by the Fund for Scientific Research Flanders, FWO project G.0125.08, and by the Special Research Fund of the University of Antwerp BOF NOI UA 2007.

†Volatility can be seen as a measure of how heavily the asset price fluctuates.
by a Brownian motion with constant volatility. Now this volatility will itself be allowed to behave randomly. Different assumptions about the random behavior of the volatility define different stochastic volatility models. A commonly used stochastic volatility model is the Heston model [10]. In chapter 3 we will present a method to price options for the Heston model within the path integral framework. Also the Heston model is not perfect, for example in this model the interest rate is assumed to be constant over time. In chapter 3 the Heston model is extended with a random interest rate, that is assumed to follow a Cox-Ingersoll-Ross process [11]. Furthermore in this chapter we demonstrate how the path integral approach can be used to determine the option price in the Heston model extended with a stochastic interest rate.

Figure 1.1: This figure shows the time evolution of the S&P 500 index (left) and the US dollar/British pound foreign exchange rate (right). The inset shows in both cases the logreturn which equals \( \ln \left( \frac{S_i}{S_{i-1}} \right) \) where \( S \) is the underlying asset price. The S&P data came from Standard and Poor’s web site: http://www.standardandpoors.com/home/en/us. The foreign exchange data came from http://www.oanda.com/currency/historical-rates.

Stochastic volatility models can take in to account a price dynamics with a rather smooth varying fluctuation size. However these models don’t generate sudden large changes in the price or in the volatility. To produce these characteristics jump models enter the scene. In chapter 5 we discuss how option pricing formulas for stochastic volatility models can be generalized to be valid in the setting where these stochastic volatility models can also have sudden large changes in the asset price. Illustrating this method we derive approximate option pricing formulas for the stochastic volatility model, where the volatility is governed by an exponential Vašíček model. These results are then generalized to the situation where the asset price can also experience large sudden movements. We focus on jumps guided by the Kou and the Merton model [12,13].

Until now we only considered the most elementary option contract. However option contracts can come in many different flavors. In this work we will also pay attention to the so called Asian option. The price of an Asian option contract will depend on the average of the asset price during a future time interval. For example the holder of an Asian option contract might have the right to buy the underlying
asset at a future time $T$ for the average price between now and $T$. We will focus on the standard geometric and arithmetic Asian option. In chapter 4 the pricing of a geometric Asian option in the Black, Scholes and Merton model using the path integral framework is discussed. This framework is suitable in this case because the payoff of the option is a functional of the entire path rather than the value at the final time. In chapter 6 we will treat the pricing of geometric and arithmetic Asian options within Lévy models. Lévy models can, for the moment, be seen as a generalization of Brownian motion. In section 2.2.3 it will be defined what they are exactly. For the Lévy models considered in chapter 6 the price of the geometric Asian option can still be determined with standard techniques. The derivation of closed form pricing formulas for the arithmetic Asian option turns out to be to difficult in these models. Instead we generalize the existing upper and lower bound for the Arithmetic Asian option price in the Black, Scholes and Merton model to the framework of Lévy models.

All these models have parameters that need to be determined. What data can be used to calibrate these models? Probably the most straightforward approach is to gather the relevant historical information and then try to determine its statistics. However the general belief is that the statistics from the past will not persist into the future. Therefore the most popular approach is to infer the future volatility of asset price movements from observed option prices and the future interest dynamics from bond prices. The general opinion is that future price movements cannot be predicted from forward and future\footnote{Bonds, forwards and futures are explained in section 2.2.} contracts [14], although there are papers that are not convinced of this [15]. Concerning the future volatility the debate is still ongoing whether historical information or observed option price data or both should be used to calibrate the models. The general opinion is that observed option price data contains more information than the historical asset price data (for more information see [16–22], where [22] is a nice review paper). For more information on the calibration of interest rates we refer to [23–25].

Since we want to have a better understanding of the calibration problem, we entered this research area. At first the goal was to study characteristics of asset prices implied by option prices. We were for example interested in the relationship between the interest rate implied by bond prices the drift of the asset implied by future prices and the interest rate and drift implied by option prices. The problem was that in our opinion not enough research has been done to test the methods for extracting information from option prices. Therefore, in chapter 7, we turn our attention to the examination of the performance of these methods. We consider two popular methods for inferring the option implied density: the so called double lognormal approach and the method based on smoothing the volatility smile (see Fig. 2.2). Additionally we construct our own method. For these three methods we investigate how well they could reproduce the densities corresponding to different models from their option prices. Furthermore we investigate how robust the methods are with respect to random errors.

In sections 2.1 and 2.3 I will discuss the necessary mathematics for understanding this thesis. Rather than giving a general introduction to the different topics, I will try to illuminate them with the help of examples which are used in the following
sections and chapters. I hope in this way to have minor overlap with the general introductions that can be found in standard textbooks and at the same time to give enough information to understand the things to come.

1.1 Contracts

In this section we will define the financial contracts used in this work.

A **forward contract** is an agreement to buy or sell an asset at a predetermined future time for a certain price. The contract is traded over the counter, it is a unstandardized private contract between two parties. The contract is usually held until its maturity time and then settled. A private contract settled at the end of its life implies some credit risk.

A **futures contract** is basically the same as a forward contract, it is also an agreement to buy or sell an asset at a predetermined future time for a certain price. The details make the two contracts substantially different from each other. Futures are standardized contracts traded on an exchange. Futures are settled daily\(^3\) and usually closed out before maturity. The presence of an exchange results in practically no credit risk.

A **bond** with a principal of €100, a maturity of \(n\) years and a coupon of \(d\%\) per year paid for example semiannually produces every half year a sum of \(\frac{d}{2}\) and at the end of the \(n\) years a sum of \(€100 + \frac{d}{2}\). The value of a bond price is primarily influenced by the expectation of the future interest rate evolution.

An **option contract** gives the holder the right to buy or to sell an asset for a certain price and with a certain maturity date. When the owner has the right to buy, the option is called a call option. When he has the right to sell it is called a put option. When he has the right to exercise the option at any time before the maturity date it is called an American option. If the option can only be exercised at the maturity date it is called a European option. Unless otherwise stated options throughout this work are European style. If there are no further specifications to the option contract, then it is also called a vanilla option.

**Asian options** are exotic options for which the payoff depends on the average price of the underlying asset during the lifetime of the option \([26], [27], [28], [29]\). One distinguishes between **average price** and **average strike** Asian options. The payoff of an average price is given by \(\max(\bar{S}_T - K, 0)\) and \(\max(K - \bar{S}_T, 0)\) for a call and put option respectively. Here \(K\) is the strike price and \(\bar{S}_T\) denotes the average price of the underlying asset at maturity \(T\). \(\bar{S}_T\) can either be the arithmetical or geometrical average of the asset price. Average price Asian options cost less than plain vanilla options. They are useful in protecting

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\(^3\)Every day it is checked whether the value of the contract has in- or decreased. If it increased the owner of the contract might be able to redraw money otherwise he will have to deposit some money.
the owner from sudden short-lasting price changes in the market for example due to order imbalances [30]. Average strike options are characterized by the following payoffs: \( \max(S_T - \bar{S}_T, 0) \) and \( \max(\bar{S}_T - S_T, 0) \) for a call and put option respectively, where \( S_T \) is the price of the underlying asset at maturity \( T \).

**Discrete or continuous?** In practice contracts are what is called discretely monitored. For example for the Asian option a set of dates \( \{t_1, ..., t_N\} \) is specified and the average asset price is determined by \( \frac{1}{N} \sum_{k=1}^{N} S_{t_k} \). In literature continuous monitoring is often assumed. The average asset price is then given by:

\[
\frac{1}{T} \int_0^T S(t) dt.
\]

With forwards futures and options the person who agrees to buy is said to have a long position, the one who agrees to sell a short position.

### 1.2 What has been done, and how we contribute

In this section we will discuss what has been done concerning the topics covered in this thesis. We will start with the empirical studies which have been done. Then we will look at models more complex than the Black-Scholes model stimulated by these empirical studies. First we will consider the stochastic volatility models, then the Lévy models, followed by a combination of these two models and then we will mention some other important models. After the models we will discuss the literature on Asian options. After reviewing the literature we will indicate how we have contributed to it. We will try to collect the most influential papers concerning stochastic volatility. Almost for sure we will be unsuccessful, and important contributions are likely to be overlooked, we apologize for this. Furthermore we will explain why these developments are important and how we contribute.

#### 1.2.1 Empirical studies

Empirical studies have as a goal to determine the most important properties of asset price dynamics (the so called stylized facts). This leads to requirements which an asset price model should fulfill. Several aspects of the dynamics of the asset price have been studied. Below some aspects are mentioned, for a complete survey we refer to the following books and papers [4, 6, 7, 31–36].

The shape of the distribution of price changes is examined by several authors [31, 32, 37–43]. For example Gopikrishnan et al. [40] find an asymptotic power law behavior for the distribution of price changes and Jansen, Longin and co-workers [42, 43] study in detail the behavior of extreme stock market returns, an account on the skewness of the distribution of price returns can be found in Refs. [44–46]. Another important aspect of the dynamics of prices is the presence of autocorrelation. For all practical purposes it is safe to say that in liquid markets the autocorrelation function of price movements is zero [31, 47, 48]. Though the autocorrelation function of the absolute value of the price movements decreases only slowly with time [5, 49–52].

All these studies clearly indicate that the Black-Scholes model is insufficient to model asset prices adequately. The dynamics of the geometric Brownian motion
is much too poor to reflect the rich dynamics of asset prices. This motivates the construction of more complicated asset price models.

### 1.2.2 Stochastic volatility

Financial data, see for example Fig. 1.1, indicate that asset prices fluctuate more heavily in some time intervals than they do in others. An important concept used in finance to cope with this observation is stochastic volatility (SV). To start with we refer to the following excellent books and review papers regarding this topic [53–60].

The importance of a time-varying volatility has been known for a long time [61–64]. From around 1985 continuous time SV models have been an important part of the mathematical finance literature. A major concern is how to estimate and test these models. Computationally demanding methods, constructed during the 1990s, have lead to the improvement of some methods and the rejection of others.

In the early days a popular approach to model time-varying volatility was through a time-changed Brownian motion [65–67]. Sometimes this time-change is modeled as a deterministic function of characteristics such as trading frequency [68–71]. The first suggesting a model that results in volatility clustering is Taylor [72, 73]. Here a model in discrete time is constructed where the volatility itself is a stochastic variable.

Afterwards the majority of SV models are in continuous time. Pioneering authors on this topic are Johnson, Wiggins and Hull [74–76]. The promise of being able to generate smiles and skews [76–78], make these models very popular. In general, when considering SV models, option prices need to be calculated using approximations or numerical methods [79–82].

Stein and Stein [83] derive analytic formulas for the option price when the volatility is modeled by an Gaussian Ornstein-Uhlenbeck process. Heston [10] constructed an analytic option pricing formula for the model, which is now known as the Heston model. Here the volatility is modeled by a mean reverting square root process. For the class of non-Gaussian Ornstein-Uhlenbeck models suggested by Barndorff [84] solutions are discussed by Nicolato [85]. Generalizations of the results of Stein and Stein and Heston to a so called affine class of models are constructed by Duffie and collaborators [82,86].

Exotic options have been studied extensively in the framework of stochastic volatility models. Refs. [87–91] discuss the pricing of barrier options, methods for valuing lookback options can be found in Refs. [92,93], for American options various approaches are available, see for example Refs. [94–98].
1.2.3 Lévy models

Lévy models are another important class of models. Apart from the specialized papers mentioned later we refer to the following books and papers for a general introduction [99–108]. Lévy processes typically come in two flavors. Flavor number one are the jump-diffusion models. The underlying idea is that infrequently events occur which have a great impact on the price. Apart from these events the price evolves as a regular diffusion. At unpredictable times, relatively large, up or downward jumps are added to this regular diffusion representing the unlikely events. It is straightforward to simulate such processes with Monte Carlo methods and use this to price exotic options. Two common examples of this type of Lévy models are the Kou [13], and the Merton model [12].

Flavor number two are the infinite activity models. These models have infinitely many jumps in every time interval. These jumps can be sufficient to model the time evolution of the asset price, it is not necessary to add a Brownian motion component to this dynamics [109]. These models are supposed to model the asset price better at all relevant time scales [109, 110]. Models without diffusion component with a finite activity [111] are said not to lead to a realistic price dynamics [106].

An intuitively appealing approach to construct Lévy processes is as a time changed Brownian motion. This time change is interpreted as a business time [112]. Two infinite activity models constructed in this way are the variance gamma model [113–116] and the normal inverse Gaussian model [117–119].

Another approach to construct Lévy processes, introduced by Ref. [120], is as a tempered stable process. A well known member of this class is the CGMY model studied by Carr et al. [109]. These processes are also studied under the name of KoBoL processes by Boyarchenko [121]. Other applications concerning financial modeling are presented by Cont, Matacz and co-authors [122,123].

Yet another method to construct Lévy models is to specify the probability density function of the increments directly. Probably the most well known example of this approach is the generalized hyperbolic model. This model can cover a wide variety of shapes. The parameter values can be chosen in such a way that the model equals the normal inverse gaussian or the variance gamma model and the normal distribution can be obtained as a limiting case. The generalized hyperbolic model was introduced by Barndorff [124] and applied to financial modeling by several authors [125–128].

Pricing formulas for Lévy models are derived for European options [116,129,130] and for several exotic options. Results concerning Barrier options can be found in Refs. [130–133], formulas for lookback options are presented in Refs. [130, 134, 135] and for American options results can be found in Refs. [136–140].

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This note is devoted to the ambiguity in terminology between different research communities. If you don’t have a particular definition of Lévy processes in mind already, feel free to skip this note, it might cause confusion. Concerning Lévy processes we will follow the terminology used in the mathematical finance community. In this community a Lévy process is defined by definition 3 and the particular α-stable Lévy process which possesses scale invariance and self-similarity properties is called a Lévy flight. In physics the term Lévy process is used for what the mathematical finance community calls a Lévy flight and the term “cadlag process with i.i.d. increments” is used instead of Lévy process.
1.2.4 Stochastic volatility jump diffusion models

Since stochastic volatility and Lévy models each attribute different aspects of the observed dynamics of market prices, it is obvious to combine these two approaches. Cont and Tankov [106] and Gatheral [141] motivate that the combination of jumps in returns and SV makes it possible to calibrate the implied volatility surface, without using time dependent parameters. Jumps make it possible to reproduce strong skews and smiles in the implied volatility at short maturities while SV provides for the calibration of the term structure, especially for long-term smiles. Ref. [142] concludes that any reasonably descriptive continuous-time model for equity-index returns should include both discrete jumps as well as stochastic volatility. Ref. [143] finds that integrating jumps and stochastic volatility is important for pricing derivatives. Ref. [144] extends the Heston model with jumps and discusses the pricing of American options in this model.

1.2.5 Other models

The main purpose of this paragraph is to give an idea of the abundance of asset price models. An important class of models are the ARCH-like models. Here ARCH [145] stands for autoregressive conditional heteroscedasticity. ARCH-like are all the models such as GARCH [146] (generalized autoregressive conditional heteroscedasticity), NGARCH [147] (Nonlinear GARCH), NARCH [148] (Nonlinear ARCH), etc. It would lead us to far to explain what all these models mean exactly. The main idea behind all these models is that they address the memory effects present in price fluctuation. These models are constructed in discrete time and the next movement of the volatility depends on all or a part of the previous movements of the asset price and its volatility. All these different ARCH-like models specify a different part of the history on which the next volatility movement will depend and a different functional dependence on this history.

Several studies have as a purpose to develop a theoretical model covering the observed stylized facts [149–158]. For example, with agent based models [149–152] the operations and interactions of numerous agents is simulated in an attempt to mimic and predict the dynamics of market prices. Simple heuristic decision-making rules are used to determine the behavior of individual agents. During time the agents may learn and adapt their rules.

1.2.6 Asian options

As already mentioned Asian options depend on the average price of the underlying during a certain time interval. Asian options are less expensive than their vanilla counterpart. For the needs of some investors Asian options might be more appropriate. Suppose the investor expects to receive a cash flow spread evenly over a certain time span. This investor might be interested in an Asian type option.

In the Black-Scholes setting Asian options are studied extensively. For the geometric Asian option an easy to use closed form solution exists, see chapter 4 and Refs. [28,159]. For arithmetic Asian options a closed form solution of this simplicity does not exist. As a consequence several trails have been investigated. Among
the numerical methods the most efficient ones are probably the PDE methods, see for example Refs. [160–164]. There also exists a variety of cutting edge analytical techniques to find exact or approximate solutions, for more information see Refs. [29, 165–171] for the continuous case and Refs. [161, 172–176] for the discrete case. These analytical techniques have their drawbacks, for example they can be slow to evaluate or, (when it concerns an approximation) the error might not be known. Fruitful alternatives are methods to calculate exact, fast-to-evaluate bounds. For the lower bound in the Black-Scholes model we refer to Curran and Rogers. [160, 177]. Since these lower bounds are very accurate most of the work thereafter concentrates on the upper bound, see for example Refs. [170, 171, 176, 178–180] and references therein.

These methods concentrate on the Black-Scholes model. However as noted in the previous sections this model does not meet all the needs of financial markets. For stochastic volatility models less results are available. An approach to price Asian options based on PDE methods can be found in [181], one based on simulation methods in [182], semi-analytical pricing formulas are by means of a perturbation method in Ref. [183]. Also for Lévy models less results (compared to the Black-Scholes setting) for discrete Asian options are available. Fusai derives [184] closed form solutions for geometrically averaged Asian options and a numerical scheme to price arithmetically averaged Asian options is presented. Albrecher derives [185] analytic approximations and bounds for the Esscher price of Asian options within the NIG model. In Ref. [186] lower bounds for arithmetic Asian options are calculated for the VG model.

1.2.7 How we contribute

In the previous paragraphs we have seen that empirical studies demonstrate convincingly that the Black-Scholes model is insufficient to represent market prices. We have also seen that more realistic models are excessively available. Nevertheless these models are only of interest for practitioners when accurate, fast to evaluate pricing formulas and efficient calibration methods are available.

We contribute developing and illustrating techniques to construct pricing formulas for financial derivatives. Furthermore we touch the topic of extracting information from market data. In the following paragraphs it will be described in more detail how we contribute to the existing work described above.

In chapter 3 we first derive closed form formulas for the Heston model using the path integral framework. Since similar pricing formulas are already derived [10] this contribution should mainly be seen as a theoretical illustration of how the path integral framework can be useful for pricing derivatives, hoping to inspire innovations. In the second part of this chapter a formula (Eq. (3.41)) is derived to price European vanilla options in the Heston model extended with a stochastic interest rate, modeled by a Cox-Ingersoll-Ross process. This result is not only theoretically interesting since a pricing formula for this model has not been derived yet. This is closely related to the work of In’t Hout and co-workers [187] where the Heston stochastic volatility model is extended with a Hull-White model for the interest rate.
In chapter 4 we study the pricing of geometrically averaged Asian options with path integrals. Prices for both average price and average strike Asian options are derived. The formulas we derive for these two option types have the same level of complexity as the Black-Scholes formula for a vanilla option. Again not only the derived formulas are important but also the techniques which are used are essential.

In chapter 5 we contribute to the research concerning stochastic volatility models extended with jumps in the asset price. To start with we present a rather general result. This result shows that if a formula for the option price in a stochastic volatility model can be derived, then this formula can be extended to give the price of an option in a model which also contains jumps in the asset price. We then apply this general result to the Heston and the exponential Vašicek model. To both these stochastic volatility models we add jumps in the asset price. We consider the Kou model and the Merton model to describe these jumps. For the Heston model the application of our general result leads to the same result as was found by Sepp [188], confirming our general result. For the exponential Vašicek model an expression for the option price was not yet available. Therefore we first derive an approximate result using path integrals for the option price in the model without jumps. Afterwards we extend this formula for the option price to be valid for the model containing jumps.

In chapter 6 we return to the problem of Asian options. Only this time the problem is made more interesting at three points. First, we consider the more relevant arithmetic Asian options. Second, we take into consideration that asset prices can only be observed at a discrete set of times, only these dates can be used for calculating the average. Third, a much more general dynamics for the asset price is taken into account. More precisely a general result will be derived for those Lévy processes for which an explicit expression for the characteristic function of the increments of the logreturn is available. For this class of models we will derive closed form expressions for a lower and an upper bound of the price of a discretely monitored arithmetic Asian option. These general results will be explicitly evaluated for the Kou model, the Merton model, the variance gamma model, the normal inverse Gaussian model and the CGMY model. These bounds are compared with existing bounds and an existing numerical method.

In chapter 7 we consider the more or less inverse problem of extracting information from observed option prices. This time we have as input a set of European vanilla option prices with the same maturity but with different strikes. Theoretically it is possible to construct the probability distribution of the asset price which leads to this set of option prices. However this theoretical statement only holds when we dispose over an option price for every strike price. This requirement is obviously unfulfilled in practice. Our last contribution is to investigate to what extent it is possible to determine the underlying asset price distribution, if we only have a finite number of option prices at our disposal. We construct a method to do this ourselves and we compare this with two methods used in literature.

Finally there are two general remarks. First, all our option pricing formulas are checked with Monte Carlo simulations. Second, the final formula for some option price is not the only thing that matters here. The methods used to derive these results are at least equally important. Furthermore these methods should be seen
in the general context of diffusion problems.

1.2.8 Critical note

Suppose you start with the Black-Scholes model and observe that market prices exhibit stochastic volatility. As a consequence you add stochastic volatility to your model. Instead of two parameters (if you are pricing risk neutrally) your model now has six parameters (with the Heston model). Furthermore you notice that sometimes large sudden changes in the asset price occur and therefore you add a jump process to your stochastic volatility model. Now your model has ten parameters (maybe more depending on how you model these jumps and whether you add correlation between the jump process and the other random processes) instead of six. This might not be the end, because you might want to improve your model even further, for example to take jumps in the volatility into account. Though suppose you are satisfied with your ten-parameter model. Now you will use this model to price options. Therefore you first need to calibrate this model. Moreover, since you realize that your model is still not perfect you will recalibrate it often. Presumably you want to price several options on several underlying assets. For each underlying a different set of parameters is necessary. So, obviously you need a fast routine to determine these ten parameters. Very likely, you will not find the “optimal” parameters. Even more there is a possibility that the prices you find will be further away from the “real” prices than those you found with the Black-Scholes model (for which you did find the optimal parameters).

Another concern when using complex models, is that the available data might not be sufficient to distinguish between two complex models. For example how can you know for sure whether prices are following a finite activity Lévy process or an infinite activity Lévy process, if prices only change at discrete times? The choice between two models will therefore be highly influenced by the availability of closed form formulas, how straightforward it is to implement a Monte Carlo simulation and so on.
Chapter 2

Mathematical preliminaries

This chapter is devoted to the necessary mathematics that is needed to understand this work. We will start with what can be seen as the basic tools of modeling diffusion processes, in particular in the context of mathematical finance. These basic tools are Wiener processes, the Itô theorem and the Kolmogorov equation. These tools will allow us to discuss the Black-Scholes model and stochastic volatility models. For Lévy models, discussed subsequently, the concept of Lévy processes is set forth briefly, the important though technical Lévy-Itô decomposition is included in the appendix. Section 2.3 illustrates the complete procedure of constructing the probability distribution at time $T$ of a stochastic process knowing the initial value. This derivation will make use of the path integral framework.

2.1 Wiener processes, the Itô theorem and the Kolmogorov equation

The financial quantities (assets) we have in mind in this work have an uncertain dynamics, they are said to follow a stochastic process. It is impossible to model this dynamics in its full complexity, assumptions need to be made. A very common assumption is that future price movements only depend on the present value of the asset, not on the entire or a part of the history of the asset price. This is called the Markov property. Although obvious criticism can be made towards this property we will also make this assumption throughout this work. One can consider discrete time models, which means that the asset price can only change at fixed dates, or one can consider continuous models where the asset price can change value at any time. The basic building block for constructing stochastic processes is the Wiener process or the Brownian motion. For a mathematically satisfying definition of, and discussion on Wiener processes we refer to [189,190]. Meanwhile we content ourselves with the following definition of a Wiener process.

**Definition 1** A stochastic process $X_t$ with $t \in \mathbb{R}_+$ with continuous sample paths is called a Wiener process or a Brownian motion with drift $\mu$ and volatility $\sigma$ when for every $t_{i-1} < t_i < t_{i+1}$ the increments $X_{t_i} - X_{t_{i-1}}$ and $X_{t_{i+1}} - X_{t_i}$ are independent.
and gaussian distributed:

\[
P (X_{t_i} - X_{t_{i-1}}) = \frac{1}{\sigma \sqrt{2 \pi (t_i - t_{i-1})}} \exp \left( -\frac{(X_{t_i} - X_{t_{i-1}} - \mu (t_i - t_{i-1}))^2}{2\sigma^2 (t_i - t_{i-1})} \right).
\]

When \( \mu = 0 \) and \( \sigma = 1 \) the process is called a standard Brownian motion (often this is simply called the Brownian motion and then the process above is called the generalized Brownian motion). Now let us look at this from a more probabilistic point of view. When playing with dice we know that if we throw one dice once, the set of possible outcomes is \{1, 2, 3, 4, 5, 6\} and for every element its corresponding probability is \( \frac{1}{6} \). What is the set \( \Omega \) of all possible future realizations of a financial quantity? Well we do not know but assume it is the set of continuous functions on the interval \([0, \infty)\), \( \Omega = C[0, \infty) \). The stochastic process \( X_t \) that we use to model the asset price maps an element \( \omega \) from the set \( \Omega \) onto its value at time \( t \), \( X_t (\omega) = \omega (t) \). When we choose to model the asset price by a Brownian motion starting at \( x_0 \) the probability that an element \( \omega \in \Omega \) lies between \( a_i \) and \( b_i \) at successive times \( t_i \) for \( i = 1, 2, \ldots N \) is given by:

\[
P (a_i \leq X_{t_i} \leq b_i ; \ i = 1, 2, \ldots N) = \\
\int_{a_1}^{b_1} \cdots \int_{a_N}^{b_N} \prod_{i=1}^{N} \frac{1}{\sqrt{2 \pi \sigma^2 (t_i - t_{i-1})}} \exp \left( -\frac{(x_i - x_{i-1} - \mu (t_i - t_{i-1}))^2}{2\sigma^2 (t_i - t_{i-1})} \right) dx_1 \ldots dx_N
\]

Evidently the Wiener process itself is often not the best choice to model the asset price, for example because it can attain negative values. Asset prices are often modeled by stating how the price will evolve in an infinitesimal time. For example for the Black-Scholes model this goes as follows:

\[
dS = \mu S dt + \sigma S dW
\]  

(2.1)

The first term on the right hand side of the previous equation means that a deterministic growth rate is assumed proportional to \( S \). The \( dW \) term represents an infinitesimal step of the Brownian motion. In this model it is assumed that bigger prices have bigger fluctuations, more precisely fluctuations are assumed to be proportional to the price. This assumption also avoids that the price becomes negative. The general process determined by an expression like (2.1) is often called an Itô process.

**Definition 2** A stochastic process of the form

\[
S (t) = \int_0^t \mu (S (u), u) du + \int_0^t \sigma (S (u), u) dW (u),
\]

so, with as infinitesimal step

\[
dS = \mu (S, t) dt + \sigma (S, t) dW,
\]  

(2.2)

is called an Itô process. An equation like (2.2) is called a stochastic differential equation (SDE).
The process determined by equation (2.1) at least stays positive, the probability that $S_t$ lies between $a_i$ and $b_i$ at successive times $t_i$ for $i = 1, 2, \ldots N$ is given by:

$$P(a_i \leq S_{t_i} \leq b_i ; \; i = 1, 2, \ldots N) = \prod_{i=1}^{N} \frac{1}{S_{t_i} \sqrt{2\pi \sigma^2 (t_i - t_{i-1})}} \times \exp \left( -\frac{\left( \ln \left( \frac{S_{t_i}}{S_{t_{i-1}}} \right) \right)}{2\sigma^2 (t_i - t_{i-1})} \right) \right).$$

(2.3)

It is the Wiener process with the Gaussian distribution replaced by the lognormal distribution. That this is indeed so can be easily seen with the help of the simplified version of the Itô lemma:

**Theorem 1 (Itô lemma)** Let $X(t)$ be an Itô process and $f(x,t)$ a function with continuous partial derivatives $\frac{\partial f(x,t)}{\partial t}$, $\frac{\partial f(x,t)}{\partial x}$ and $\frac{\partial^2 f(x,t)}{\partial x^2}$. If $S = f(X,t)$ then the differential equation for $S$ is given by

$$dS = \frac{\partial f(X,t)}{\partial X} dX + \frac{\partial f(X,t)}{\partial t} dt + \frac{\partial^2 f(X,t)}{2\partial X^2} (dX)^2.$$

If we write $x(t) = \ln \left( \frac{S(t)}{S_0} \right)$ where $S(t)$ is the process determined by equation (2.1) then the differential equation for $x(t)$ is given by:

$$dx = \frac{1}{S} (\mu S dt + \sigma S dW) - \frac{1}{2S^2} (\mu S dt + \sigma S dW)^2$$

(2.4)

The expectation value of $(W(t+h) - W(t))^2$ is equal to $h$. Therefore in the limit of $h$ going to an infinitesimal $dt$, $(dW)^2$ can be replaced by $dt$, furthermore the term $dW dt$ which is of order $dt \sqrt{dt}$ and the term $(dt)^2$ do not contribute in the infinitesimal limit. As a consequence equation (2.4) becomes:

$$dx = \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dW.$$ 

(2.5)

Which is just a Brownian motion, this justifies the statement we made that the joint probability distribution (2.3) corresponds to equation (2.1).

The time evolution of the probability density corresponding to a stochastic process can be described by a partial differential equation. Common names for this equation are the Kolmogorov forward and the Fokker-Planck equation. The equation is given by the following theorem.

**Theorem 2 (Kolmogorov forward equation)** Let $X(t)$ be an $N$-dimensional Itô process controled by the following stochastic differential equation:

$$dX = \mu (X,t) dt + \sigma (X,t) dW;$$
with $\mu$ an $N$-dimensional vector, $\sigma(X,t)$ an $N \times N$ matrix and $dW$ an $N$-dimensional vector of independent Wiener processes. Then the probability distribution $P(x,t)$ is a solution of the following partial differential equation:

$$
\frac{\partial P(x,t)}{\partial t} = - \sum_{i=1}^{N} \frac{\partial \mu_i(x,t) P(x,t)}{\partial x_i} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial^2 D_{ij}(x,t) P(x,t)}{\partial x_i \partial x_j},
$$

(2.6)

with $D = \sigma(X,t) \cdot \sigma'(X,t)$ where $\sigma'$ is the transposed matrix of $\sigma$.

For example, the probability density corresponding to the stochastic process governed by equation (2.5) is a solution of the partial differential equation

$$
\frac{\partial P(x,t)}{\partial t} = - \mu \frac{\partial P(x,t)}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 P(x,t)}{\partial x^2}
$$

(2.7)

This equation is usually solved with boundary conditions $\lim_{x \to \pm \infty} P(x,t) = 0$ and initial conditions $P(X,0) = \delta(X - X_0)$, where $\delta$ is the delta function. This means that at time zero one knows exactly what the asset is worth. This might be disputed and instead of starting with a delta function as distribution one might want to start with a distribution having a nonzero variance.

2.2 Finance

2.2.1 Black-Scholes-Merton model

Figure 2.1: The left panel shows a possible realization of the asset price within the Black-Scholes model. The right panel shows the corresponding logreturn. As parameters we took $\sigma = 0.2$, $T = 5$, $r = \mu = 0.0367$.

The benchmark option pricing theory is still the one developed by Black, Scholes and Merton [1, 2]. Pricing options starts by assuming a certain dynamics for the underlying asset. In this model, as already mentioned, the asset price obeys the following stochastic differential equation

$$
dS = \mu Sdt + \sigma SdW
$$

(2.8)
A possible evolution of the asset price in this model is illustrated in Fig. 2.1. It is seen that equation (2.8) results in a quite regular kind of random behavior. Especially if you look at the logreturn, you see that it has the same behavior for all times. In the Black-Scholes setting the common approach is to determine the so-called risk-free price. This is also how we will proceed. To this end the Itô lemma is used to infer that the option price $V(S, t)$ itself obeys the SDE given by:

$$dV = \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial t} dt + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} dt$$

(2.9)

Next a portfolio containing a short position in one option and $\frac{\partial V}{\partial S}$ shares of the underlying is constructed. The value $\Pi$ of this portfolio is then equal to $-V + \frac{\partial V}{\partial S} S$. The change in the value of this portfolio in an infinitesimal time step is given by:

$$d\Pi = -dV + \frac{\partial V}{\partial S} dS$$

If expression (2.9) for $dV$ is substituted in the above equation one obtains:

$$d\Pi = - \left( \frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} \right) dt$$

(2.10)

Since there is no term containing stochastic uncertainty in the above equation one argues that the gain of this portfolio should equal the gain one obtains from investing at the risk free interest rate, i.e. $d\Pi = r\Pi dt$. Substituting this in expression (2.10) one obtains the following equation:

$$\frac{\partial V}{\partial t} = -rS \frac{\partial V}{\partial S} - \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + rV$$

(2.11)

This equation is then solved backwards with as final condition the payoff function of the option. For example, for a call option this is $\max(S_T - K, 0)$ and the boundary conditions for the call option price $C$ are $C(0, t) = 0$ and $\lim_{S \to \infty} \frac{C(S,t)}{S} = 1$. The solution of this equation for a call option, which can be found in almost every textbook about option pricing (see for example [26]), is given by:

$$C = S_0 N \left( \frac{\ln \left( \frac{S_0}{K} \right) + \left( r + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right) - Ke^{-rT} N \left( \frac{\ln \left( \frac{S_0}{K} \right) + \left( r - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right)$$

(2.12)

where $N$ represents the normal cumulative distribution function. The above derivation together with all the assumptions that were implicitly made can be found in numerous textbooks for example in [26]. It is also possible to follow an other approach to find the option price. One then tries to determine the probability distribution function of the asset price at the maturity time $P(S, T)$ and uses $P(S, T)$ to calculate the expectation value of the option’s payoff function. The expected gain

*It is very common to assume that it is not possible to make a risk free profit larger than with an investment at the risk free interest rate. This is called a no-arbitrage argument.
then needs to be discounted to its present value. According to this argumentation the call option price is given by

\[ e^{-rT} \int_0^\infty \max (S - K, 0) \, P(S, T) \, dS \]  

(2.13)

The distribution \( P(S, T) \) which solves the forward Kolmogorov equation (2.7) is known to be:

\[ P(S, T) = \frac{1}{\sigma S \sqrt{2\pi T}} \exp \left( -\frac{\left( \ln \left( \frac{S}{S_0} \right) - \left( \mu - \frac{\sigma^2}{2} \right) T \right)^2}{2\sigma^2 T} \right) \]

Substituting this into equation (2.13) gives the following expression for the call price:

\[ C = e^{-rT} \left[ S_0 e^{\mu T} N \left( \frac{\ln \left( \frac{S_0}{K} \right) + \left( \mu + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right) - K N \left( \frac{\ln \left( \frac{S_0}{K} \right) + \left( \mu - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right) \right] \]

(2.14)

For people unfamiliar with this research field it might seem strange that there are two different formulas (2.12) and (2.14) for the same object. One might think one of them must be wrong. They are not wrong, but will be used in different situations. For example you might be very convinced that the underlying asset will have a rate of return \( \mu \) and use equation (2.14) to value the call option. You are taking a risk then (except if you have more information than the other market participants), since no asset can have a risk-free rate of return higher than the risk free interest rate. On the other hand you might dislike taking risk, then you construct a risk-free portfolio such as in the derivation above and constantly re-balance it in such a way that it remains risk less at all times. The initial price you will ask for the option (so no gains or losses are made) is then given by expression (2.12). This is called risk neutral valuation and is far more popular than the other approach.

If the interest rate that should be used in formula (2.12) can somehow be determined then the volatility is the only remaining parameter in this formula and there is a one to one correspondence between option prices and volatilities. Volatilities corresponding to observed market prices of options are called implied volatilities. If the Black and Scholes theory was correct then the implied volatility would be constant as a function of time and maturity, this is known to be false, see Fig. 2.2. When the implied volatility looks more or less like a symmetric upward parabola it is called a volatility smile.

It is well known that the pioneering option pricing theory of Black and Scholes [1] and Merton [2] fails to reflect some important empirical phenomena. Many studies have been conducted to modify and improve the Black-Scholes model. Among others, popular models include, (a) the local volatility models [191]; (b) the stochastic volatility (SV) models [10, 76, 83]; (c) the SV and stochastic interest rate models [147, 192–194]; (d) the jump diffusion models [12, 13, 82, 195]; (e) models based on Lévy process [104–106, 196, 197]; and (f) the SV jump diffusion models [142–144, 188, 198–200].
Figure 2.2: This figure shows volatilities implied by S&P 500 data for three different maturities. The starting date is 1/7/09, the value of the index at that time was 923.33, as interest rate we used $r = 0.03$. The data used here came from http://www.historicaloptiondata.com.

2.2.2 Heston model

Figure 2.3: The left panel shows a possible realization of the asset price within the Heston model. The middle panel shows the corresponding logreturn and the right panel the corresponding variance. As parameters we took $\sigma = 0.18$, $T = 5$, $r = \mu = 0.0367$, $\nu n = 0.04$, $\kappa = 0.5$, $\theta = 0.04$, $\sigma = 0.14$, $\rho = -0.4$.

Stochastic volatility models are popular extensions of the Black-Scholes model. Their motivation is straightforward, the volatility of most assets is not constant and not deterministic. Therefore an extra stochastic differential equation is introduced to describe its dynamics. A lot of different equations for this dynamics can be found in the literature. Here we will focus on the Heston model. The stochastic processes $S(t)$ and $v(t)$ of the Heston model are determined by the following equations:

\begin{align*}
    dS &= \mu S dt + S \sqrt{v} dw_1, \\
    dv &= \kappa (\theta - v) dt + \sigma \sqrt{v} \left( \rho dw_1 + \sqrt{1 - \rho^2} dw_2 \right).
\end{align*}

(2.15) \quad (2.16)

Where $dw_1$ and $dw_2$ are independent standard Wiener processes. When $v$ is smaller than $\theta$ the term $\kappa (\theta - v) dt$ is positive, when it is bigger the term is negative, so that in both cases it pushes $v$ back into the direction of $\theta$. Models with this property are
called mean reverting. $\theta$ is called the mean reversion level and $\kappa$ the mean reversion rate. The parameter $\rho$ controls the correlation between the random movement of $S$ and the random movement of $v$. A possible realization of the asset price, its log return and the variance in this model are shown in Fig. 2.3. This model can clearly mimic a more complex dynamics. There are regions where the variance is high which results in big fluctuations in the asset price and there are regions where the variance is low resulting in small fluctuations. This phenomenon is known as volatility clustering and is also seen in real data, see Fig. 1.1.

The derivation of the partial differential equation for the option price, this is the extension of equation (2.11) to the Heston framework, is more involved. Also in this situation one will try to set up a risk-free portfolio. Though to this end one has to be able to trade in volatility risk. We return to this topic in chapter 3. For all the details we refer to [10].

2.2.3 Lévy models

A straightforward generalization of the Wiener process is the Lévy process. If the condition that the increments are normally distributed and the left continuity condition are omitted from the definition of the Wiener process we arrive at the Lèvy process.

**Definition 3 (Lévy process)** A stochastic process $X_t$ with $t \in [0, T]$ which is right continuous with left limits: for each $t \in [0, T]$ the limit $\lim_{s \to t, s < t} X(s)$ exists and $\lim_{s \to t, s > t} X(s) = X(t)$, is called a Lévy process if for every $t_i-1 < t_i < t_{i+1}$ the increments $X_{t_i} - X_{t_{i-1}}$ and $X_{t_{i+1}} - X_{t_i}$ are independent and identically distributed (i. i. d.) and $\forall \varepsilon > 0, \lim_{h \to 0} P(|X(t+h) - X(t)| \geq \varepsilon) = 0$.

The last condition is not always included in the definition of a Lévy process, it makes sure we only have random jumps. Predictable jumps are not included in the model. There exists an intuitively appealing characterization of Lévy processes. A first concept needed to arrive there is the Poisson process.

**Definition 4 (Poisson process)** Let $T_m = \sum_{i=1}^{m} s_i$ with $s_i$ a sequence of independent exponentially distributed random variables with parameter $\lambda$, which means that $s_i$ is positive and its probability distribution is given by $\lambda e^{-\lambda s_i}$. The process $N_t$ defined by

$N_t = \sum_{m \geq t \geq T_m} e^{-\lambda s_m}$

is called a Poisson process with intensity $\lambda$.

In the definition above $s_i$ can be interpreted as the time between two random events. The Poisson process $N_t$ then counts how many random events have already occurred at time $t$. The Poisson process will here be used as tool to construct jump processes. The process $N_t$ controls the number of jumps that have occurred up to
time $t$. Then, on average $\lambda t$ jumps have occurred up to time $t$. The size of these jumps can be a random variable itself. A jump process constructed this way is also known as a compound Poisson process and is defined as follows.

**Definition 5 (Compound Poisson process)** Let $N_t$ be a Poisson process with intensity $\lambda$ and let $J_i$ be independent and identically distributed random variables independent from $N_t$ with distribution $P_J$. Then the stochastic process $X_t$ defined as

$$X_t = \sum_{i=1}^{N_t} J_i$$

is called a compound Poisson process with intensity $\lambda$ and jump size distribution $P_J$.

The reason that these compound Poisson processes are introduced here is because every Lévy process $X_t$ allows the following decomposition:

$$X_t = \mu (X, t) + \sigma (X, t) W_t + X_1^t + X_2^t. \tag{2.17}$$

Here the first two terms on the left hand side form an Itô process. The term $X_1^t$ is a compound Poisson process. The term $X_2^t$ can be interpreted as a process having infinitely many jumps in every finite time interval. Most of these jumps are small though. This decomposition is called the Lévy-Itô decomposition. More information on this decomposition and what the term $X_2^t$ means exactly can be found in appendix 2.A.

When this last term is absent the Lévy process $X_t$ is a so called jump diffusion. Most of the time such a jump diffusion behaves exactly like a regular diffusion. Though sometimes a relatively large upward or downward jump is added to the asset price. These can be interpreted as the occurrence of some rare event. The behavior of these jumps can be independently tuned from the regular diffusion part. It can be chosen how often on average a jump should occur. Furthermore the distribution of the jump size can be chosen. The Merton [12] and the Kou [13] model used later in this work are typical examples of jump diffusions.

When the term $X_2^t$ is present $X_t$ is called an infinite activity jump process. With these models the regular diffusion term and the $X_1^t$ term is often omitted. Because even without these terms the time evolution of the process is sufficiently extensive to mimic market data, possibly even better than jump diffusions [109, 110]. On the other hand these models can be more difficult to simulate. The normal inverse Gaussian [118], the variance gamma [113] and the CGMY [109] belong to this class of Lévy models.

The method of constructing a risk free portfolio introduced in the Black-Scholes setting, was already a bit tricky in the Heston framework. With Lévy models it is impossible to hedge against jump risk. A common approach to price options in these models is to require that the discounted bond price is a martingale. We come back to this topic in chapter 6.

To conclude this short introduction on Lévy models, the Kou model which is a typical example of a jump model will be discussed. In this model the dynamics of the logreturn $x$ is given by:

$$dx = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW + dX^t_1.$$
Here the first two terms on the right hand side of the equation are the same as in the Black-Scholes model. The last term $dX^j_t$ represents the possible jump of the compound Poisson process $X^j_t$ at time $t$. For the Kou model the jump size distribution consist of two exponential distributions, one for the up jumps with parameter $\eta_1$, and one for the down jumps with parameter $\eta_2$. The up jump distribution has weight $p$, and the down jump distribution weight $1 - p$. This way if there is a jump the probability of it being an up jump can be different from it being a down jump. Fig. 2.4 illustrates the dynamics of the asset price in this model. This model results in a Black-Scholes dynamics with isolated large jumps.

![Figure 2.4: In this figure several aspects of the Kou model are illustrated. In the left panel the jump size distribution is shown. In the middle panel a possible realization of the asset price is shown. The logreturn corresponding to this asset price evolution is shown in the right panel. Here the following parameter values were used: $\sigma = 0.120381$, $r = 0.0367$, $\lambda = 1.3$, $p = 0.3$, $\eta_1 = 0.02$, $\eta_2 = 0.05$.](image)

### 2.3 Deriving the propagator of the Cox-Ingersoll-Ross model

In this section the process of deriving a propagator corresponding to a stochastic differential equation is illustrated. This derivation will make use of the path integral framework. Fine introductions to the theory of path integrals as well as the treatment of advanced topics can be found in Refs. [201–207]. Instead of rewording such an introduction, the determination of the transition probability corresponding to the Cox-Ingersoll-Ross (CIR) model will be discussed in detail. To determine this transition probability the path integral corresponding to the radial harmonic oscillator will be solved. To start with we will outline the procedure.

1. We start from the stochastic differential equation of the CIR process, Eq. (2.18).

2. The first task is to derive the infinitesimal propagator corresponding to this stochastic differential equation.

3. Proposition 3 relates the infinitesimal propagator to the forward Kolmogorov equation.
4. The forward Kolmogorov equation corresponding to Eq. (2.18), given by theorem 2, is not of the form of proposition 3. A transformation of variables leads to a Kolmogorov equation of the form of proposition 3, the infinitesimal propagator is given by Eq. (2.28).

5. This infinitesimal propagator allows us to construct the path integral for the CIR process, Eq. (2.30).

6. This path integral finally leads to the desired propagator of the CIR process, Eq. (2.35).

2.3.1 The infinitesimal propagator and the Kolmogorov equation

In the CIR model the dynamics of the interest rate \( r \) is governed by the following stochastic differential equation

\[
\mathrm{d}r = \kappa (\theta - r) \, \mathrm{d}t + \sigma \sqrt{r} \, \mathrm{d}w.
\] (2.18)

This is the same stochastic differential equation that is used in the Heston model to describe the time evolution of the volatility. Given that the interest rate at \( t = 0 \) equals \( r_0 \) we want to know the probability distribution \( P (r_T, T | r_0, 0) \) of \( r_T \) at a certain time \( T \). This is also called a transition probability and has the following general property:

\[
P (r_T, T | r_0, 0) = \int_0^\infty P (r_T, T | r_s, s) \, P (r_s, s | r_0, 0) \, \mathrm{d}r_s.
\]

In the same line of thoughts it is seen that also the following holds:

\[
P (r_T, T | r_0, 0) = \lim_{N \to \infty} \int \cdots \int \prod_{i=1}^N P (r_i, t_i | r_{i-1}, t_{i-1}) \, \mathrm{d}r_1 \cdots \mathrm{d}r_{N-1},
\] (2.19)

with \( r_N = r_T, t_N = T, t_0 = 0 \). We take this limit because for an infinitesimal time step the transition probability \( P \) can often be simplified to \( P_\varepsilon \) and the following equality holds:

\[
P (r_T, T | r_0, 0) = \lim_{N \to \infty} \int \cdots \int \prod_{i=1}^N P_\varepsilon (r_i, t_i | r_{i-1}, t_{i-1}) \, \mathrm{d}r_1 \cdots \mathrm{d}r_{N-1}.
\] (2.20)

The next theorem relates this infinitesimal time transition probability \( P_\varepsilon \) to the forward Kolmogorov equation.

**Proposition 3** The infinitesimal time propagator corresponding to the forward Kolmogorov equation

\[
\frac{\partial P (r, t)}{\partial t} = \left( \frac{1}{2} \frac{\partial V_1 (r, t)}{\partial r} + V_2 (r, t) \right) P (r, t) - \frac{\partial V_1 (r, t) P (r, t)}{\partial r} + \frac{\sigma^2}{2} \frac{\partial^2 P (r, t)}{\partial r^2}
\] (2.21)
The following equality holds:

\[ P_\varepsilon (r_i, t_i \mid r_{i-1}, t_{i-1}) = \frac{1}{\sqrt{2\pi v_\varepsilon}} \exp \left( -\frac{(r_i - r_{i-1} - V_1 \left( \frac{t_i + r_{i-1} - t_i}{2} \right) t_{i-1} - t_i)^2}{2v_\varepsilon} \right). \] (2.22)

**Proof.** This will be proved by showing that the infinitesimal time propagator (2.22) gives rise to the forward Kolmogorov equation (2.21). To get rid of all the subscripts, \( x, y, t \) and \( \varepsilon \) will be used instead of \( r_i, r_{i-1}, t_i \) and \( (t_{i-1} - t_i) \). Assuming expression (2.22) the following holds:

\[ P (x, t + \varepsilon) = \int_0^{+\infty} P_\varepsilon (x, t + \varepsilon \mid y, t) P (y, t) \, dy \]

\[ = \frac{1}{\sqrt{2\pi v_\varepsilon}} \int_0^{+\infty} \exp \left( -\frac{(x - y - V_1 \left( \frac{x + y}{2} \right) \varepsilon)^2}{2v_\varepsilon} + V_2 \left( \frac{x + y}{2}, t \right) \varepsilon \right) P (y, t) \, dy. \]

Since we are interested in the infinitesimal time behavior we will expand both sides of this equation around \( \varepsilon = 0 \). The left hand side then becomes

\[ P (x, t + \varepsilon) = P (x, t) + \frac{\partial P (x, t)}{\partial t}. \] (2.23)

To expand the right hand side the substitution \( y = x - \delta \) is applied, which results in

\[ = \frac{1}{\sqrt{2\pi v_\varepsilon}} \int_{-\infty}^{x} \exp \left( -\frac{(\delta - V_1 (x - \frac{\delta}{2}, t) \varepsilon)^2}{2v_\varepsilon} + V_2 \left( x - \frac{\delta}{2}, t \right) \varepsilon \right) P (x - \delta, t) \, d\delta \]

\[ = \frac{1}{\sqrt{2\pi v_\varepsilon}} \int_{-\infty}^{x} \exp \left( -\frac{(x - \delta)^2}{2v_\varepsilon} + \frac{\delta V_1 (x - \frac{\delta}{2}, t)}{v} \varepsilon + V_2 \left( x - \frac{\delta}{2}, t \right) \varepsilon \right) P (x - \delta, t) \, d\delta. \] (2.24)

Let us use for a moment the following notation

\[ f (\delta, \varepsilon) = \exp \left( \frac{\delta V_1 (x - \frac{\delta}{2}, t)}{v} - \frac{V_1^2 (x - \frac{\delta}{2}, t) \varepsilon}{2v} + V_2 \left( x - \frac{\delta}{2}, t \right) \varepsilon \right) P (x - \delta, t). \]

If in equation (2.24), \( f (\delta, \varepsilon) \) is replaced by its Taylor expansion we obtain

\[ = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{(i+j)!} \frac{d^{i+j} f (\delta, \varepsilon)}{d\varepsilon^i d\delta^j} \Bigg|_{(0,0)} \varepsilon^i \delta^j \frac{1}{\sqrt{2\pi v_\varepsilon}} \int_{-\infty}^{x} \exp \left( -\frac{(x - \delta)^2}{2v_\varepsilon} \right) \, d\delta. \] (2.25)
For small $\varepsilon$ the following asymptotic equality holds

$$
\frac{1}{\sqrt{2\pi \nu \varepsilon}} \int_{-\infty}^{+\infty} \exp \left( -\frac{\delta^2}{2\nu \varepsilon} \right) \delta^i d\delta = \exp \left( -\frac{x^2}{2\nu \varepsilon} \right) \sum_i c_i \varepsilon^i,
$$

with $c_i$ some constants. Therefore the integral in the above expression is negligible in the small time limit and up to order $\varepsilon$ the following equality holds

$$
\frac{1}{\sqrt{2\pi \nu \varepsilon}} \int_{-\infty}^{x} \exp \left( -\frac{\delta^2}{2\nu \varepsilon} \right) \delta^i d\delta = \frac{1}{\sqrt{2\pi \nu \varepsilon}} \int_{-\infty}^{+\infty} \exp \left( -\frac{\delta^2}{2\nu \varepsilon} \right) \delta^i d\delta
$$

The latter integral has as solution 0 when $j$ is odd and $(\nu \varepsilon)^{\frac{j}{2}}$ when $j$ is even, with $(j-1)!! = (j-1)(j-3)(j-5)\ldots 1$. Up to order $\varepsilon$ equation (2.25) becomes:

$$
f (0,0) + \frac{d f (\delta, \varepsilon)}{d \varepsilon} \bigg|_{(0,0)} \varepsilon + \frac{1}{2} \frac{d^2 f (\delta, \varepsilon)}{d \delta^2} \bigg|_{(0,0)} \varepsilon,
$$

which equals

$$
P (x, t) + \left( -\frac{V_1^2 (x, t)}{2\nu} + V_2 (x, t) \right) P (x, t) \varepsilon +
$$

$$
\left. \left( \frac{V_1 (x-\frac{\varepsilon t}{2})}{v} + \frac{\delta^2}{4 v^2} V_1 (x-\frac{\varepsilon t}{2}) - \frac{\delta^2}{4 v^2} V_1 (x-\frac{\varepsilon t}{2}) \varepsilon + \frac{dV_2 (x-\frac{\varepsilon t}{2})}{d\delta} \varepsilon \right)^2 \right|_{v \varepsilon}
$$

$$
P (x - \delta, t) +
$$

$$
\left. \left( \frac{\delta^2}{4 v^2} V_1 (x-\frac{\varepsilon t}{2}) + \frac{\delta^2}{4 v^2} V_1 (x-\frac{\varepsilon t}{2}) \varepsilon + \frac{dV_2 (x-\frac{\varepsilon t}{2})}{d\delta} \varepsilon \right) \right|_{v \varepsilon}
$$

$$
P (x - \delta, t) +
$$

$$
\left. \left( \frac{dV_2 (x-\frac{\varepsilon t}{2})}{d\delta} + \frac{d^2 P (x-\delta, t)}{d\delta^2} \right) \right|_{v \varepsilon}
$$

$$
(2.26)
$$

$$
= P (x, t) + \left( -\frac{V_1^2 (x, t)}{2\nu} + V_2 (x, t) \right) P (x, t) \varepsilon +
$$

$$
\frac{1}{2} \left( \frac{V_1 (x, t)}{v} \right)^2 P (x, t) - \frac{1}{v} \frac{d}{dx} V_1 (x, t) P (x, t) - \frac{dV_1 (x, t)}{v} \frac{dP (x, t)}{dx} + \frac{d^2 P (x, t)}{dx^2} \varepsilon
$$

$$
(2.27)
$$

Here we used $\frac{d}{d\delta} P (x - \delta, t) = -\frac{d}{dx} P (x - \delta, t)$ and $\frac{d}{d\delta} V_1 (x - \frac{\delta t}{2}) = -\frac{d}{dx} V_1 (x - \frac{\delta t}{2})$.

Since expression (2.27) equals the right hand side of expression 2.23, the following equality holds

$$
\frac{\partial P (x, t)}{\partial t} = \left( \frac{1}{2} \frac{\partial V_1 (x, t)}{\partial x} + V_2 (x, t) \right) P (x, t) - \frac{\partial V_1 (x, t) P (x, t)}{\partial x} + \frac{v \partial^2 P (x, t)}{2 \partial x^2}.
$$

To come to a forward Kolmogorov equation which is of the form of (2.21) we do the substitution $r = z^2$. The Itô lemma gives us the stochastic differential equation for $z$.

$$
dz = \frac{1}{2} \left( \kappa \left( \frac{\theta}{z} - z \right) - \frac{\sigma^2}{4z} \right) dt + \frac{\sigma}{2} dw
$$
Using Theorem (2), expression (2.6), we find that the forward Kolmogorov equation corresponding to this differential equation is given by

\[
\frac{\partial P (z, t)}{\partial t} = - \frac{1}{2} \left( \kappa (\frac{\theta}{\bar{z}} z - \bar{z}) - \frac{\sigma^2}{4} \right) P (z, t) + \frac{\sigma^2}{8} \frac{\partial^2 P (z, t)}{\partial z^2}.
\]

Proposition 3 learns that the infinitesimal propagator corresponding to this Kolmogorov equation is given by:

\[P_{\varepsilon} (x, t + \varepsilon | y, t) = \frac{1}{\sigma \sqrt{2\pi \varepsilon}} \exp \left( - \frac{2(x-y - \frac{1}{2} \kappa (\frac{\theta}{\bar{y}} - \frac{\sigma^2}{4\bar{y}}) \varepsilon)^2}{\varepsilon} \right), \tag{2.28}\]

with \(\bar{y} = \frac{x + y}{2}\). Before proceeding the path integral will be defined.

### 2.3.2 The path integral of the radial harmonic oscillator

**Definition 6** A Wiener path integral

\[
\int_{x(0) = x_0}^{x(T) = x_T} \exp \left[ - \int_0^T f (\dot{x}(t), x(t)) \, dt \right] \, Dx(t)
\]

is defined as the following limit

\[
\lim_{N \to \infty} \int \cdots \int dx_1 \ldots dx_{N-1} A_N \exp \left[ - \sum_{i=1}^N f \left( \frac{x_i - x_{i-1}}{\varepsilon}, \frac{x_i + x_{i-1}}{2} \right) \varepsilon \right],
\]

with \(\varepsilon = \frac{1}{N}\), \(x_N = x_T\) and \(A\) the proper normalization factor.

Notice that when the integral \(\int_0^T f (\dot{x}(t), x(t)) \, dt\) is discretized it is evaluated in the point \(\frac{x + x_{i-1}}{2}\). For a standard Riemann integral it does not matter in which point between \(x_i\) and \(x_{i-1}\) we evaluate the discretized integral. Does it matter here? To answer this question we go back to the proof of proposition 3. Suppose that in this proof the functions \(V_1\) and \(V_2\) are evaluated in the point \(y\). After the substitution \(y = x - \delta\), \(V_1\) becomes \(V_i (x - \delta, t)\) with \(i = 1, 2\). But then in expression (2.26) the derivatives \(\frac{d}{d\delta} V_i (x - \frac{\delta}{2}, t)\) are replaced by \(\frac{d}{d\delta} V_i (x - \delta, t)\). For \(V_2\) this does not matter because its derivatives always come together with higher order terms in \(\varepsilon\) and are therefore negligible. For \(V_1\) on the other hand there are relevant terms \(\frac{d}{d\delta} V_i (x - \delta, t)\), replacing these by \(\frac{d}{d\delta} V_i (x - \delta, t)\) in expression (2.26) we would eventually obtain the following forward Kolmogorov equation:

\[
\frac{\partial P (x, t)}{\partial t} = V_2 (x, t) P (x, t) - \frac{\partial V_1 (x, t) P (x, t)}{\partial x} + \frac{v^2}{2} \frac{\partial^2 P (x, t)}{\partial x^2}.
\]

26
To put it differently the propagator \( P (x_T, T \mid x_0, 0) \) equals both

\[
\lim_{N \to \infty} \int \cdots \int dx_1 \ldots dx_{N-1} \left( \frac{1}{\sigma \sqrt{2\pi}} \right)^N \times \\
\exp \left[ \sum_{i=1}^{N} -2 \left( x_i - x_{i-1} - \frac{1}{2} \left( \kappa \left( \frac{\theta}{\bar{y}} - \bar{y} \right) - \frac{\sigma^2}{4\bar{y}^2} \right) \varepsilon \right)^2 \right] \\
\exp \left[ \sum_{i=1}^{N} -2 \left( x_i - x_{i-1} - \frac{1}{2} \left( \kappa \left( \frac{\theta}{\bar{y}} - \bar{y} \right) - \frac{\sigma^2}{4\bar{y}^2} \right) \varepsilon \right)^2 \right],
\]

(2.29)

with \( \bar{y} = \frac{x_i + x_{i-1}}{2} \) and

\[
\lim_{N \to \infty} \int \cdots \int dx_1 \ldots dx_{N-1} \left( \frac{1}{\sigma \sqrt{2\pi}} \right)^N \times \\
\exp \left[ \sum_{i=1}^{N} -2 \left( x_i - x_{i-1} - \frac{1}{2} \left( \kappa \left( \frac{\theta}{x_i - x_{i-1}} \right) - \frac{\sigma^2}{4(x_i - x_{i-1})} \right) \varepsilon \right)^2 \right].
\]

The reason we choose for the midpoint alternative (2.29) to define the path integral is because then the usual rules such as

\[
\int_{0}^{T} \dot{g}(x) \, dt = g(x_T) - g(x_0).
\]

are applicable. This we will not prove, more information can be found in the references at the beginning of this section. Remember that the discretization problem only mattered for the \( V_1 \) term, for the \( V_2 \) it makes no difference whether we discretize it by \( V_2 (x_i), \frac{\nu_2(x_i) + \nu_2(x_{i-1})}{\nu_2 \left( \frac{x_i + x_{i-1}}{2} \right)}, \) etc. So we arrived at the point where the propagator \( P (z_T, T \mid z_0, 0) \) for the CIR process can be expressed in terms of a path integral.

\[
P (z_T, T \mid z_0, 0) = \int_{z(0)=z_0}^{z(T)=z_T} \exp \left[ \int_{0}^{T} \left( -2 \left( z_i - \frac{1}{2} \left( \kappa (\frac{\theta}{z_i} - \frac{\sigma^2}{4z_i^2}) \right) \right)^2 \right) \right] Dz(t) \quad (2.30)
\]

Taking all the terms with the same \( z \) dependence together we arrive at:

\[
P (z_T, T \mid z_0, 0) = \exp \left( \frac{2\kappa\theta}{\sigma^2} - \frac{1}{2} \right) \ln \left( \frac{z_T}{z_0} \right) - \frac{\kappa}{\sigma^2} \left( z_T - z_0 \right) + \frac{\kappa^2\theta}{\sigma^2} T \right) \times \\
\int_{z(0)=z_0}^{z(T)=z_T} \exp \left[ \int_{0}^{T} -2 \frac{\dot{z}^2}{\sigma^2} - \frac{\kappa^2 z^2}{2\sigma^2} + \left( \frac{\kappa\theta}{2} - \frac{\kappa^2\theta^2}{2\sigma^2} - \frac{\sigma^2}{32} - \frac{\sigma^2}{16} \right) \frac{1}{z^2} \right] Dz(t) \quad (2.31)
\]

Introducing the notations \( a = \frac{\sigma^2}{T}, b = \frac{\sigma}{2}, \lambda = 2\frac{\nu^2 \theta}{\sigma^2} - 1 \) simplifies the remaining calculations. The path integral that still has to be solved

\[
\int_{z(0)=z_0}^{z(T)=z_T} \exp \left[ - \int_{0}^{T} \left( \frac{\dot{z}^2}{2a} + \frac{\left( \lambda^2 - \frac{1}{4} \right) a}{2z^2} + \frac{b^2z^2}{2a} \right) \right] Dz(t), \quad (2.32)
\]

27
is known as the one dimensional radial harmonic path integral. Although this path integral is treated in Refs. [202, 207] it is not easy to find a treatment of this path integral accessible for beings unfamiliar with path integrals. To find the solution of the path integral given by expression (2.32) we first need some preliminary lemmas.

**Lemma 4** In the limit of $z$ going to infinity the asymptotic approximation holds for the modified Bessel function of the first kind and order $\lambda$

$$I_\lambda (z) = \frac{1}{\sqrt{2\pi z}} \exp \left[ z - \left( \frac{\lambda^2 - \frac{1}{4}}{2z} \right) \right] \left( 1 + O \left( \frac{1}{z^2} \right) \right).$$

**Proof.** In ref [208] at page 203 we find the following asymptotic expansion for $I_\lambda (z)$:

$$I_\lambda (z) = \frac{\exp (z)}{\sqrt{2\pi z}} \sum_{m=0}^{\infty} \frac{(-)^m (\lambda, m)}{(2z)^m} + \frac{\exp (-z + (\lambda - \frac{1}{2}) \pi i)}{\sqrt{2\pi z}} \sum_{m=0}^{\infty} \frac{(\lambda, m)}{(2z)^m}, \quad (2.33)$$

where $(\lambda, m)$ equals

$$(\lambda, m) = (-)^m \left( \frac{1}{2} - \lambda \right) m \left( \frac{1}{2} + \lambda \right) m, \quad \text{with } (z)_m = z(z+1)(z+2)\ldots(z+m-1) \text{ the Pochhammer symbol.}$$

When $z$ approaches infinity the second summation becomes negligible with respect to the first one. Furthermore the first summation can be written as:

$$I_\lambda (z) = \frac{\exp (z)}{\sqrt{2\pi z}} \left( 1 + \frac{(\frac{1}{2} - \lambda)^2}{2z} + O \left( \frac{1}{z^2} \right) \right)$$

$$= \frac{\exp (z)}{\sqrt{2\pi z}} \left( 1 + O \left( \frac{1}{z^2} \right) \right).$$

**Lemma 5** In the limit of $\varepsilon$ going to zero the following equality holds:

$$\frac{1}{\sqrt{2\pi a \varepsilon}} \exp \left[ -\frac{(z_i - z_{i-1})^2}{2a \varepsilon} - \frac{(\lambda^2 - \frac{1}{4}) a}{2z_i z_{i-1} \varepsilon} - \frac{b^2 (z_i^2 + z_{i-1}^2)}{4a \varepsilon} \right] \left[ 1 + O \left( \varepsilon^2 \right) \right]$$

$$= \frac{\sqrt{z_i z_{i-1} b}}{a \sinh (b \varepsilon)} \exp \left( -\frac{b (z_i^2 + z_{i-1}^2)}{2a \coth (b \varepsilon)} \right) I_\lambda \left( \frac{z_i z_{i-1} b}{a \sinh (b \varepsilon)} \right).$$

**Proof.** If $\varepsilon$ goes to zero the argument of the Bessel function in the latter equation reaches infinity and lemma 4 can be used with the following result for the right hand
Proof. On page 718 of Ref. [209], Eq. 6.633.4, one can find the following equality:

$$\int_0^\infty \frac{\sqrt{z_1 z_0 b}}{a \sinh (b \varepsilon)} \exp \left( - \frac{b(z_1^2 + z_0^2)}{2a} \coth (b \varepsilon) \right) I_\lambda \left( \frac{z_1 z_0 b}{a \sinh (b \varepsilon)} \right) \, dz_1 = \int_0^\infty \frac{\sqrt{z_2 z_0 b}}{a \sinh (b \varepsilon)} \exp \left( - \frac{b(z_2^2 + z_0^2)}{2a} \coth (b \varepsilon) \right) I_\lambda \left( \frac{z_1 z_0 b}{a \sinh (b \varepsilon)} \right) \, dz_2.$$ (2.34)

Lemma 6

Proof. On page 718 of Ref. [209], Eq. 6.633.4, one can find the following equality:

$$\int_0^\infty z \exp \left( -az^2 \right) I_\lambda (bz) J_\lambda (cz) \, dz = \frac{1}{2a} \exp \left( \frac{b^2 - c^2}{4a} \right) J_\lambda \left( \frac{bc}{2a} \right),$$

with Re[a] > 0 and Re[\lambda] > -1. Changing c to ic and multiplying both sides by $$e^{-\frac{ict}{2}}$$ the above equality becomes:

$$\int_0^\infty z \exp \left( -az^2 \right) I_\lambda (bz) I_\lambda (cz) \, dz = \frac{1}{2a} \exp \left( \frac{b^2 + c^2}{4a} \right) I_\lambda \left( \frac{bc}{2a} \right).$$
With the help of this equality the expression at the left hand side of equation (2.34) simplified as follows:

\[
\int_0^\infty \frac{\sqrt{z_1} z_0 b}{a \sinh (b \varepsilon)} \exp \left( -\frac{b}{2a} \left( z_1^2 + z_0^2 \right) \coth (b \varepsilon) \right) I_\lambda \left( \frac{z_1 z_0 b}{a \sinh (b \varepsilon)} \right) \frac{\sqrt{z_1} z_2 b}{a \sinh (b \varepsilon)} \times \exp \left( -\frac{b}{2a} \left( z_2^2 + z_0^2 \right) \coth (b \varepsilon) \right) I_\lambda \left( \frac{z_1 z_2 b}{a \sinh (b \varepsilon)} \right) dz_1
\]

\[
= \frac{\sqrt{z_2} z_0 b}{a^2 \sinh (b \varepsilon) \sinh (b \varepsilon)} \frac{1}{2a} \left( \coth (b \varepsilon) + \coth (b \varepsilon) \right) I_\lambda \left( \frac{z_1 b}{a \sinh (b \varepsilon)} \frac{z_2 b}{a \sinh (b \varepsilon)} \right)
\]

\[
\times \exp \left( -\frac{b}{2a} \left( z_0^2 \coth (b \varepsilon) + z_2^2 \coth (b \varepsilon) \right) + \left( \frac{z_1 b}{a \sinh (b \varepsilon)} \right)^2 + \left( \frac{z_2 b}{a \sinh (b \varepsilon)} \right)^2 \right)
\]

\[
= \frac{\sqrt{z_2} z_0 b}{a \left[ \cosh (b \varepsilon) \sinh (b \varepsilon) + \sinh (b \varepsilon) \cosh (b \varepsilon) \right]}
\]

\[
\times I_\lambda \left( \frac{z_2 z_0 b}{a \left[ \cosh (b \varepsilon) \sinh (b \varepsilon) + \sinh (b \varepsilon) \cosh (b \varepsilon) \right]} \right)
\]

\[
\times \exp \left( \frac{b}{2a} \left( \frac{z_0^2}{\sinh^2 (b \varepsilon) \cosh (b \varepsilon) \coth (b \varepsilon) + \sinh (b \varepsilon) \cosh (b \varepsilon) \coth (b \varepsilon)} + \frac{z_2^2}{\sinh^2 (b \varepsilon) \cosh (b \varepsilon) \coth (b \varepsilon) + \sinh (b \varepsilon) \cosh (b \varepsilon) \coth (b \varepsilon)} \right) \right)
\]

\[
= \frac{\sqrt{z_2} z_0 b}{a \sinh (b (l + 1) \varepsilon)} I_\lambda \left( \frac{z_2 z_0 b}{a \sinh (b (l + 1) \varepsilon)} \right)
\]

\[
\times \exp \left( \frac{b}{2a} \left( \frac{(1 - \cosh (b \varepsilon) \sinh (b \varepsilon) + \cosh (b \varepsilon) \sinh (b \varepsilon)) z_0^2}{\sinh (b \varepsilon) \cosh (b \varepsilon) + \sinh (b \varepsilon) \cosh (b \varepsilon) \coth (b \varepsilon)} \right) \right)
\]

\[
= \frac{\sqrt{z_2} z_0 b}{a \sinh (b (l + 1) \varepsilon)} I_\lambda \left( \frac{z_2 z_0 b}{a \sinh (b (l + 1) \varepsilon)} \right)
\]

\[
\times \exp \left( \frac{b}{2a} \left( \frac{(1 - \cosh (b \varepsilon) \sinh (b \varepsilon) + \cosh (b \varepsilon) \sinh (b \varepsilon)) z_2^2}{\sinh (b \varepsilon) \cosh (b \varepsilon) + \sinh (b \varepsilon) \cosh (b \varepsilon) \coth (b \varepsilon)} \right) \right)
\]

\[
= \frac{\sqrt{z_2} z_0 b}{a \sinh (b (l + 1) \varepsilon)} \exp \left( -\frac{b}{2a} \left( z_1^2 + z_0^2 \right) \coth (b (l + 1) \varepsilon) \right) I_\lambda \left( \frac{z_2 z_0 b}{a \sinh (b (l + 1) \varepsilon)} \right).
\]

We are now ready to prove the proposition which solves the path integral given by expression (2.32).
Proposition 7  The solution of the path integral

\[ z(T) = z_T \]

\[
\int_{z(0) = z_0} \exp \left[ - \int_0^T \left( \frac{\dot{z}^2}{2a} + \frac{(\lambda^2 - \frac{1}{4})a}{2z^2} + \frac{b^2 z^2}{2a} \right) dt \right] Dz(t)
\]

is given by

\[
\frac{\sqrt{z_T z_0 b}}{a \sinh (bT)} \exp \left( - \frac{b(z_T^2 + z_0^2)}{2a} \coth (bT) \right) I_\lambda \left( \frac{z_T z_0 b}{a \sinh (bT)} \right)
\]

Proof. According to definition 6 the path integral given above is equal to:

\[
\lim_{N \to \infty} \int \cdots \int dz_1 \cdots dz_{N-1} \prod_{i=1}^N \frac{1}{\sqrt{2\pi a \varepsilon}} \exp \left[ \left( \frac{(z_i - z_{i-1})^2}{2a \varepsilon} - \frac{(\lambda^2 - \frac{1}{4}) a \varepsilon}{2z_i z_{i-1}} - \frac{b^2 (z_i^2 + z_{i-1}^2)}{4a \varepsilon} \right) \right].
\]

According to lemma 5 this is equal to

\[
\lim_{N \to \infty} \int \cdots \int dz_1 \cdots dz_{N-1} \prod_{i=1}^N \frac{\sqrt{z_i z_{i-1} b}}{a \sinh (b \varepsilon)} \exp \left( - \frac{b(z_i^2 + z_{i-1}^2)}{2a} \coth (b \varepsilon) \right) I_\lambda \left( \frac{z_i z_{i-1} b}{a \sinh (b \varepsilon)} \right).
\]

Lemma 6 allows one to carry out the integrations one by one. After carrying out the first integral we obtain:

\[
\lim_{N \to \infty} \int \cdots \int dz_2 \cdots dz_{N-1} \prod_{i=3}^N \frac{\sqrt{z_i z_{i-1} b}}{a \sinh (b \varepsilon)} \exp \left( - \frac{b(z_i^2 + z_{i-1}^2)}{2a} \coth (b \varepsilon) \right) I_\lambda \left( \frac{z_i z_{i-1} b}{a \sinh (b \varepsilon)} \right) \times I_\lambda \left( \frac{z_2 z_1 b}{a \sinh (2b \varepsilon)} \right) \prod_{i=3}^N \frac{\sqrt{z_i z_{i-1} b}}{a \sinh (b \varepsilon)} \exp \left( - \frac{b(z_i^2 + z_{i-1}^2)}{2a} \coth (b \varepsilon) \right) I_\lambda \left( \frac{z_i z_{i-1} b}{a \sinh (b \varepsilon)} \right).
\]

After performing the other integrations we end up with:

\[
\lim_{N \to \infty} \frac{\sqrt{z_T z_0 b}}{a \sinh (N \varepsilon)} \exp \left( - \frac{b(z_T^2 + z_0^2)}{2a} \coth (N \varepsilon) \right) I_\lambda \left( \frac{z_T z_0 b}{a \sinh (N \varepsilon)} \right).
\]

Since \( N \varepsilon = T \) the latter function does not depend on \( N \) and the limit is equal to:

\[
\frac{\sqrt{z_T z_0 b}}{a \sinh (bT)} \exp \left( - \frac{b(z_T^2 + z_0^2)}{2a} \coth (bT) \right) I_\lambda \left( \frac{z_T z_0 b}{a \sinh (bT)} \right).
\]

Substituting this expression in to (2.31) and returning to the original variable \( r \) with parameters \( \kappa, \theta \) and \( \sigma \) we finally obtain the solution for \( P(r_T, T \mid r_0, 0) \).

\[
P(r_T, T \mid r_0, 0) = \frac{\kappa (r_T r_0)^{\frac{2\theta}{4a^2 \sinh (\frac{\theta T}{2})}}}{4a^2 \sinh (\frac{\theta T}{2})} I_2 \left( \frac{2\kappa \sqrt{r_T r_0}}{\sigma^2 \sinh (\frac{\theta T}{2})} \right)
\]

\[
\times \exp \left( \frac{\kappa^2 \theta}{\sigma^2} T - \frac{\kappa}{\sigma^2} (r_T - r_0) - \frac{\kappa (r_T + r_0)}{\sigma^2} \coth (\frac{\kappa T}{2}) \right)
\]

(2.35)
2.A Lévy-Itô decomposition

Here some more details concerning the decomposition of a Lévy process into a sum of a regular diffusion a jump process and an infinite activity Lévy process will be given. Since the Lévy-Itô decomposition is actually a rather involved theorem we will not even here go into full detail, for this we refer to [106] and references therein. Nevertheless we want the reader to be able to catch the main idea of this theorem without having to run to the library. Even to explain the theorem superficially it is useful to introduce some concepts.

Let \( X_t \) be a compound Poisson process with intensity \( \lambda \) and jump size distribution \( P_{js} \). With this process one can associate what is called a Poisson random measure on \([0, T] \times \mathbb{R}^n\). For a set \([s, t] \times A \subset [0, T] \times \mathbb{R}^n\) this measure, denoted by \( J_X \), counts how many jumps occurred in the interval \([s, t]\) with jump sizes in \( A \), formally with \( B \subset [0, T] \times \mathbb{R}^n \)

\[
J_X(B) = \# \{(t, X_t - X_{t-}) \in B\},
\]

with \( X_{t-} = \lim_{s \to t, s \lt t} X(s) \). For \( A \subset \mathbb{R}^n \) fixed, \( J_X(\cdot, A) \) is a Poisson process with intensity \( \lambda P_{js}(A) \), since it counts the number of jumps with sizes in \( A \). Furthermore the compound Poisson process can be seen as an expectation value with respect to this measure:

\[
\int_{[0,t] \times \mathbb{R}^n} xJ_X(ds \times dx) = \sum_{s \in [0,t] \cap X_s \neq X_{s-}} \Delta X_s = X_t
\]

If the construction of the measure \( J_X \) is generalized we arrive at the formal definition of the Poisson random measure.

**Definition 7 (Poisson random measure)** Let \( E \subseteq \mathbb{R}^n \) and \( \mathcal{B} \) the Borel \( \sigma \)-algebra on \( E \) and \( \mu \) a measure on \((E, \mathcal{B})\) such that for every compact measurable set \( B \in \mathcal{B} \), \( \mu(B) < \infty \). Let \((\Omega, \mathcal{F}, P)\) be a probability space. A measure

\[
M : \Omega \times \mathcal{B} \to \mathbb{N}
\]

such that

- For almost all \( \omega \in \Omega \), \( M(\omega, \cdot) \) is a measure on \( E \) such that for every compact measurable set \( B \in \mathcal{B} \), \( M(\omega, B) < \infty \).

- For each \( \mu \)-measurable set \( A \subset E \), \( M(\cdot, A) \) is a Poisson random variable with parameter \( \mu(A) \):

- The variables \( M(A_1), \ldots, M(A_n) \) are independent.

is called a Poisson random measure on \( E \) with intensity measure \( \mu \).

This measure is called random because it is a mapping from \( \Omega \times \mathcal{B} \) to \( \mathbb{N} \), a genuine measure would be a mapping from \( \mathcal{F} \times \mathcal{B} \) to \( \mathbb{N} \). In the example of the compound Poisson process an \( \omega \in \Omega \) represents a possible sequence of jump times, \( E = [0, T] \times \mathbb{R}^n \) and \( \mu([s_1, s_2] \times A) = \lambda P_{js}(A)(s_2 - s_1) \). Of course we are not only interested in the concept of a compound Poisson process but also in the general Lévy process. To this end the important notion of a Lévy measure is introduced.
Definition 8 (Lévy measure) Let \((X_t)_{t \geq 0}\) be a Lévy process on \(\mathbb{R}^n\) and \(A \in \mathcal{B}(\mathbb{R}^n)\) the Borel sigma algebra on \(\mathbb{R}^n\). A measure \(\nu\) on \(\mathbb{R}^n\) defined by:
\[
\nu(A) = E[\# \{ t \in [0, 1] : \Delta X_t \neq 0, \Delta X_t \in A \}]
\]
is called the Lévy measure of \(X\).

In words, \(\nu(A)\) is the expected number of jumps whose size belongs to \(A\), per unit time. In the situation of the compound Poisson process \(\nu(A) = \lambda P_j s(A)\). Finally the Lévy-Itô decomposition can be stated more precisely.

Theorem 8 (Lévy-Itô decomposition) Let \((X_t)_{t \geq 0}\) be a Lévy process on \(\mathbb{R}^n\) and \(\nu\) the Lévy measure of \(X\). Then the following statements hold:

1. \(\nu\) is a measure on \(\mathbb{R}^n_0\) such that for every compact measurable set \(B \in \mathcal{B}\), \(\nu(B) < \infty\) and for which
\[
\int_{\|x\| \leq 1} \|x\|^2 \nu(dx) < \infty, \quad \int_{\|x\| \geq 1} \nu(dx) < \infty.
\]

2. \(J_X\), the jump measure of \(X\) is a Poisson random measure on \([0, T] \times \mathbb{R}^n\) with intensity measure \(\nu(\cdot) \Delta T\).

3. There exists an \(n\)-dimensional vector \(\mu\) and Wiener process \((B_t)_{t \geq 0}\) such that
\[
X_t = \mu t + B_t + X'_t + \lim_{\varepsilon \to 0} X^\varepsilon_t,
\]
\[
X'_t = \int_{\|x\| \geq 1, s \in [0, t]} x J_X(ds \times dx),
\]
\[
X^\varepsilon_t = \int_{\|x\| \leq 1, s \in [0, t]} \left[ x [J_X(ds \times dx) - \nu(dx)ds]\right],
\]

with \(B_t\), \(X'_t\) and \(X^\varepsilon_t\) independent.

Lévy processes can have a diverging number of small jumps. The condition \(\int_{\|x\| \leq 1} \|x\|^2 \nu(dx) < \infty\) puts some bounds on the small jump behaviour. The result \(\int_{\|x\| \geq 1} \nu(dx) < \infty\) means that the number of jumps larger then 1 is finite. Actually for any \(\delta\) it is true that the number of jumps larger than \(\delta\) is finite. \(X'_t\) is a Compound Poisson process and the term \(\lim_{\varepsilon \to 0} X^\varepsilon_t\) takes the possible contribution of infinitely many small jumps in to account.
Chapter 3

A path integral treatment of the Heston model with stochastic interest rate*

It is known for a long time that the BS model is only a crude approximation to the economic reality and that its assumptions are violated in actual markets. Perhaps the most illustrative violation is that the volatility implied from traded vanilla options is not constant across strikes and maturities. Examples of models that tackle such violations are local volatility processes [106, 191], jump processes [106], Lévy processes [105] and stochastic volatility models [210]. A stochastic volatility model that has been particularly successful at explaining the implied volatility smile see Fig. 2.2 in equity and foreign exchange markets is the Heston model [10]. In his seminal paper, Heston [10] derived a closed form solution for the price of a vanilla option, which enables a quick and reliable calibration to market prices, especially for liquidly traded vanilla options with maturities between 2 months and 2 years [211].

Path integral methods have already been used in the pricing of options within stochastic volatility models [212, 213] and in the related problem of non-Gaussian diffusion [214] (at the end of Sec. 3.1.1 we come back to this connection), but to the best of our knowledge no explicit option pricing formula as computationally cheap as Stein and Stein’s [83] or Heston’s formulas [10] have yet been derived using path integrals. We will show in the present chapter how to carry out this task for the Heston model. The result we thereby obtain corresponds to the existing result [10] for which the calibration and correspondence to market data has already been investigated (see for example [215–220]).

Afterwards we extend this result to a setting where not only the volatility but also the interest rate is stochastic and follows the widely used CIR model [11, 221–223]. To the best of our knowledge, no exact closed-form formula for this problem is available. Therefore, we have checked our formulas against a Monte Carlo simulation. Note that for the very similar problem, where the interest rate is modeled by the Hull-White model, an analytic formula is derived by ref. [187].

*This chapter is based on the article [194], which is joint work with Michiel Wouters, Jacques Tempère and Sven Foulon. Furthermore discussions with Lucien Lemmens, Ivo De Saedeleer and Karel in’t Hout are acknowledged.
The plan of the chapter is as follows. In Sec. 3.1.1, we outline our model, which is the one introduced by Heston. Further in this section we derive a closed-form solution for the time evolution of the asset price. In Sec. 3.1.2 we present a closed-form pricing formula for plain vanilla options which only involves one numerical integration of a compilation of elementary functions. In Sec. 3.2 we will extend the Heston model to include stochastic interest rate, in Sec. 3.2.1 we present a closed-form solution for the vanilla option price which still contains only on numerical integration of a compilation of elementary functions. In Sec. 3.2.2 we test this result with a Monte Carlo method and discuss the relevance of including stochastic interest rate.

3.1 Standard Heston model

3.1.1 The model and its path integral representation

We will concentrate on assets following a diffusion process described by the following two equations introduced by Heston [10]

\[ dS = \mu_0 S dt + S \sqrt{v} dw_1, \]
\[ dv = \kappa_0 (\theta_0 - v) dt + \sigma \sqrt{v} (\rho dw_1 + \sqrt{1 - \rho^2} dw_2). \]

Here \( S \) is the asset price, \( \mu_0 \) is a constant drift factor, \( v \) is the variance of the asset, \( \kappa_0 \) is the spring constant of the force that attracts the variance to its mean reversion level \( \theta_0 \) (also called the mean reversion rate), \( \sigma \) is the volatility of the variance, and \( w_1 \) and \( w_2 \) are independent Wiener processes with unit variance and zero mean. The asset price follows a Black-Scholes process [1], whereas the volatility obeys a Cox-Ingersoll-Ross process [11].

There are two general approaches to determine the price of an option in a path integral context. One could, based upon equations (3.1), (3.2) determine the probability distribution for the asset price at the strike time \( T \) conditional on the values of the asset and the variance at the present time \( P_S(S_T, v_T | S_0, v_0) \). The expectation value of the option price at time \( T \) can be calculated by integrating the gain you make with a certain outcome of \( S_T \) multiplied by the probability of obtaining that outcome \( P_S(S_T, v_T | S_0, v_0) \) over all possible values of \( S_T \). To obtain the present value of the price one then discounts this expectation value with the risk free interest rate \( r \). For a European call option this can be written as:

\[ C = e^{-rT} \int_{-\infty}^{+\infty} dS_T dv_T \max \left[ S_T - K, 0 \right] P_S(S_T, v_T | S_0, v_0). \]  

We will refer to this approach as the “asset propagation approach” since \( P_S \) is the propagator for a distribution of asset prices (and volatilities).

The other approach focuses on the option price rather than the asset evolution, as will be referred to as the “option propagation approach”. In his paper [6], Heston discusses the subtle differences between the asset point of view and the option
price point of view, and this discussion is also relevant to the present path-integral framework. Heston motivates that the time evolution of the option price $U(S,v,t)$ is governed by the following partial differential equation (pde):

$$\frac{\partial U}{\partial t} = -rS \frac{\partial U}{\partial S} + rU - \{\kappa_0 [\theta_0 - v] - \lambda v\} \frac{\partial U}{\partial v} - \frac{1}{2} \sigma^2 v \frac{\partial^2 U}{\partial S^2} - \rho \sigma v S \frac{\partial^2 U}{\partial S \partial v} - \frac{1}{2} \sigma^2 \frac{\partial^2 U}{\partial v^2},$$

(3.4)

where $\lambda$ is a parameter introduced [10] on the basis of no-arbitrage arguments and setting up a risk-free portfolio. If one makes the substitution $U = e^{rt}V$ one obtains the following PDE for $V$ as a function of the asset price and the volatility:

$$\frac{\partial V}{\partial t} = -rS \frac{\partial V}{\partial S} - \{\kappa_0 [\theta_0 - v] - \lambda v\} \frac{\partial V}{\partial v} - \frac{1}{2} \sigma^2 v \frac{\partial^2 V}{\partial S^2} - \rho \sigma v S \frac{\partial^2 V}{\partial S \partial v} - \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial v^2}.$$  

(3.5)

Based on this PDE, one can find a kernel $P_V$ that propagates a given final distribution $V(S_T, v, T)$ backwards to the present value $V(S_0, v_0, 0)$ of the option. Since the value of the option at the final time $T$ is known, $V(S_T, v, T) = e^{-rT} U(S_T, v, T) = e^{-rT} \max [S_T - K, 0]$, the value of the option now is obtained through

$$C = e^{-rT} \int_{-\infty}^{+\infty} dS_T d\nu_T \max [S_T - K, 0] P_V(S_T, \nu_T | S_0, v_0),$$

(3.6)

Furthermore the PDE (3.5) is equal to the Kolmogorov backward equation corresponding to the following system of stochastic differential equations

$$dS = rS dt + S \sqrt{\nu} dw_1,$$

(3.7)

$$dv = \{\kappa_0 [\theta_0 - v] - \lambda v\} dt + \sigma \sqrt{\nu} \left( \rho dw_1 + \sqrt{1 - \rho^2} dw_2 \right).$$

(3.8)

This means that both approaches can be dealt with simultaneously by considering a generalized stochastic process:

$$dS = \mu S dt + S \sqrt{\nu} dw_1,$$

(3.9)

$$dv = \kappa (\theta - v) dt + \sqrt{\nu} \left( \rho dw_1 + \sqrt{1 - \rho^2} dw_2 \right),$$

(3.10)

and calculating its transition probability $P(S_T, \nu_T | S_0, v_0)$. The ”asset propagation” approach (3.3) can then be retained by simply replacing $\mu, \kappa$ and $\theta$ by $\mu_0, \kappa_0$ and $\theta_0$ and the ”option propagation” approach (3.5), (3.6) by replacing $\mu, \kappa$ and $\theta$ by $r, \kappa_0 + \lambda$ and $\kappa_0 \theta_0 / (\kappa_0 + \lambda)$. The pricing formula for the European call is the same as (3.3) where this time the transition probability $P(S_T, \nu_T | S_0, v_0)$ is the one corresponding to (3.9), (3.10):

$$\begin{align*}
\left\{ 
P_S(S_T, \nu_T | S_0, v_0) &= P(S_T, \nu_T | S_0, v_0) \big|_{\mu=\mu_0, \kappa=\kappa_0, \theta=\theta_0} \\
&P_V(S_T, \nu_T | S_0, v_0) = P(S_T, \nu_T | S_0, v_0) \big|_{\mu=r, \kappa=\kappa_0 + \lambda, \theta=\kappa_0 \theta_0 / (\kappa_0 + \lambda)}
\end{align*}$$

(3.11)

We will calculate the transition density $P$ for the general stochastic process (3.9), (3.10).
For later convenience we make the following substitutions:

\[
\begin{align*}
x &= \ln \left( \frac{S}{S_0} \right) - \mu t, \\
z &= \sqrt{v},
\end{align*}
\]

Equation (3.12) becomes:

\[
\begin{align*}
\frac{dx}{dt} &= \frac{x^2}{2} - \frac{\mu}{2} t, \\
\frac{dz}{dt} &= \frac{1}{2z} \left( \kappa \theta - \frac{\sigma^2}{4} \right) - \frac{z}{2} dt + \frac{\sigma}{2} \left( \rho dw_1 + \sqrt{1 - \rho^2} dw_2 \right).
\end{align*}
\]

The substitution

\[
y(t) = x(t) - \frac{\rho}{\sigma} \left( z^2(t) - \kappa \theta t \right),
\]

leads to two uncorrelated equations:

\[
\begin{align*}
\frac{dy}{dt} &= \frac{\rho}{\sigma} \left( \frac{\kappa \theta}{\sigma} - \frac{1}{2} \right) z^2 dt + z \sqrt{1 - \rho^2} dw_1, \\
\frac{dz}{dt} &= \frac{1}{2z} \left( \kappa \theta - \frac{\sigma^2}{4} \right) - \frac{\kappa}{2} dt + \frac{\sigma}{2} dw_2.
\end{align*}
\]

The probability that \(y_T\) has the value \(y_T\) and \(z_T\) the value \(z_T\) at a later time \(T\) will be denoted as \(P(y_T, z_T | y_0, z_0)\). The advantage of transforming to these variables is that \(dw_1\) and \(dw_2\) are uncorrelated, so that the following expression holds for \(P(y_T, z_T | y_0, z_0)\):

\[
P(y_T, z_T | y_0, z_0) = \int Dz(t) \exp \left( - \int_0^T \left\{ L_Q[y(t), z(t)] + L_{CIR}[z(t)] \right\} dt \right).
\]

Where the quadratic Lagrangian \(L_Q(y(t), z(t))\) equals

\[
L_Q(y(t), z(t)) = \frac{1}{2z^2(1 - \rho^2)} \left[ \dot{y} - \left( \frac{\rho}{\sigma} \kappa - \frac{1}{2} \right) z^2 \right]^2,
\]

and the Lagrangian corresponding to the CIR process, \(L_{CIR}[z(t)]\), is given by \([224]\) and section 2.3:

\[
L_{CIR}[z] = \frac{2}{\sigma^2} \left\{ \dot{z} - \frac{1}{2} \left[ \frac{1}{z} \left( \kappa \theta - \frac{\sigma^2}{4} \right) - \kappa z \right] \right\}^2 - \frac{1}{4z^2} \left( \kappa \theta - \frac{\sigma^2}{4} \right) - \frac{\kappa}{4}.
\]

The first step in the evaluation of Eq. (3.17) is the integration over all \(y\)-paths. Because the action is quadratic in \(y\) this integration can be done analytically and yields

\[
P(y_T, z_T | y_0, z_0) = \int Dz(t) \frac{1}{\sqrt{2\pi z^2(1 - \rho^2)}} \exp \left\{ \frac{\left( \frac{\rho}{\sigma} \kappa - \frac{1}{2} \right)}{\left( 1 - \rho^2 \right)} (y_T - y_0) - \frac{1}{2(1 - \rho^2)} \left( \frac{\rho}{\sigma} \kappa - \frac{1}{2} \right)^2 z^2 - \frac{(y_T - y_0)^2}{2z^2(1 - \rho^2)} - \int_0^T dt \ L_{CIR}[z(t)] \right\}.
\]
Note that the probability to arrive in \((y_T, z_T)\) depends on the average value of the volatility along the path \(z(t): z^2 = \int_0^T z^2(t)dt,\) in agreement with Ref. [83]. However, this average value appears in the denominator of the third term, and to perform the functional integral one needs to bring this into the numerator. This is achieved by rewriting part of the expression (3.20) as follows:

\[
\frac{\exp \left[ - \frac{(y_T - y_0)^2}{2z^2(1 - \rho^2)} \right]}{\sqrt{2\pi z^2(1 - \rho^2)}} = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \exp \left[ i(y_T - y_0) k - \frac{\int z^2dt \ (1 - \rho^2) k^2}{2} \right]. \quad (3.21)
\]

Combining Eqs. (3.20) and (3.21) and making the substitution \(k = l + i\frac{(\pi - \frac{1}{2})}{(1 - \rho^2)}\) the transition probability becomes

\[
P(y_T, z_T \mid y_0, z_0) = \int_{-\infty}^{+\infty} \frac{dl}{2\pi} \exp [i(y_T - y_0) l] \int Dz(t) \times \exp \left( -\int_0^T dt \left\{ \mathcal{L}_{CIR}[z(t)] + \frac{1}{2}z^2 \left[ (1 - \rho^2) l^2 + 2li \left( \frac{l}{\sigma} \frac{\kappa}{\theta} - \frac{1}{2} \right) \right] \right\} \right). \quad (3.22)
\]

The path integral over the CIR action is formally equivalent to the exactly solvable radial harmonic oscillator (see section 2.3 and Ref. [207]) and, fortunately, adding terms proportional to \(z^2\) to the action does not spoil this equivalence. The full path integral over \(z(t)\) can be carried out without approximations with the following result:

\[
P(y_T, z_T \mid y_0, z_0) = \frac{1}{2\pi} \exp \left[ \frac{\kappa^2\theta}{\sigma^2} T + \left( 2\frac{\kappa^2}{\sigma^2} - \frac{1}{2} \right) \ln \left( \frac{z_T}{z_0} \right) - \frac{\kappa}{\sigma^2} \left( z_T^2 - z_0^2 \right) \right] \times \int_{-\infty}^{+\infty} \exp [i(y_T - y_0) l] \frac{4\omega}{\sigma^2 \sinh(\omega T)} dz_T \times \exp \left[ -\frac{2\omega}{\sigma^2} \left( z_0^2 + z_T^2 \right) \coth(\omega T) \right] \frac{4\omega z_0 z_T}{\sigma^2 \sinh(\omega T)} dl. \quad (3.23)
\]

where

\[
\omega = \frac{\sigma}{2} \sqrt{\left( \frac{\kappa}{\sigma} + il\rho \right)^2 + l(l - i)}. \quad (3.24)
\]

is the \(l\)-dependent frequency of the radial harmonic oscillator that corresponds to the CIR Lagrangian (3.19). After transforming back to the \(x\) variable we see that also the integral over the final value \(z_T\) can be done analytically (see e.g. [225]), yielding the marginal probability distribution \(P(x_T \mid 0, z_0) = \int_{-\infty}^{+\infty} dz_T P(x_T, z_T \mid 0, z_0)\) (written in the original variable \(v\)) as a simple Fourier integral:

\[
P(x_T \mid 0, v_0) = \frac{1}{2\pi} \exp \left[ \frac{\kappa}{\sigma^2} (\kappa \theta T + v_0) \right] \times \int_{-\infty}^{+\infty} N^{-\frac{2\pi}{\sigma^2} \kappa \theta} \exp \left\{ i \left[ x_T + \frac{\rho}{\sigma} (v_0 + \kappa \theta T) \right] l - \frac{2\omega}{\sigma^2 \sinh(\omega T)} [\cosh(\omega T) - N] v_0 \right\} dl, \quad (3.25)
\]

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where $N$ is:

$$N = \frac{1}{\cosh (\omega T) + \frac{1}{2\omega} (\kappa + il\rho \sigma) \sinh (\omega T)}, \quad (3.26)$$

Note the similarity of expression (3.25) with the result obtained in Ref. [214], derived for a general stochastic process with non-Gaussian noise.

### 3.1.2 Pricing of plain vanilla options

From now on we follow the option propagation approach and set $\mu$ equal to $r$. The price of a call option $C$ with expiration date $T$ and strike $K$ written in the $x$ variable is given by:

$$C = e^{-rT} \int_{-\infty}^{+\infty} dx_T \max \left[ S_0 \exp (x_T) - K, 0 \right] P (x_T \mid 0, v_0), \quad (3.27)$$

where the risk free interest rate was restored and denoted by $r$ and $P (x_T \mid 0, v_0)$ is given by Eq. (3.3). Now there are still two numerical integrations that have to be done. Following the derivation outlined in Ref. [214] we can rewrite expression (3.27) so that only one numerical integration remains:

$$C = \frac{S_0 - e^{-rT} K}{2} + i \int_{-\infty}^{+\infty} \frac{1}{l} \left\{ \exp \left[ i \left( \frac{\rho}{\sigma} a + x_e - rT \right) l + \frac{\kappa}{\sigma^2} a \right] \right. \times \left[ S_0 \exp \left( \Theta - \frac{\rho}{\sigma} a \right) - e^{-rT} K \exp (\Upsilon) \right] - S_0 + e^{-rT} K \right\} \frac{dl}{2\pi}, \quad (3.28)$$

with

$$x_e = \ln \left( \frac{K}{S_0} \right), \quad (3.29a)$$

$$a = v_0 + \kappa \theta T, \quad (3.29b)$$

$$\nu = \frac{\sigma}{2} \sqrt{\left( \frac{\kappa}{\sigma} + il\rho - \rho \right)^2 + l(l+i)}, \quad (3.29c)$$

$$M = \left[ \cosh (\nu T) + \frac{1}{2\nu} (\kappa + il\rho \sigma - \rho \sigma) \sinh (\nu T) \right]^{-1}, \quad (3.29d)$$

$$\Theta = \frac{2\nu v_0}{\sigma^2 \sinh (\nu T)} [M - \cosh (\nu T)] + \frac{2}{\sigma^2} \kappa \theta \log M, \quad (3.29e)$$

$$\Upsilon = \frac{2\nu v_0}{\sigma^2 \sinh (\omega T)} [N - \cosh (\omega T)] + \frac{2}{\sigma^2} \kappa \theta \log N. \quad (3.29f)$$

and $N$ and $\omega$ defined as before, Eqs. (3.24) and (3.26). We have tested this result against the formula stated in Ref. [10]. This confirmed the correctness of formula (3.28). Now we are confident to explore new grounds with our method in the following section.
3.2 Stochastic interest rate

3.2.1 Derivation of the option price

In the previous section we assumed the interest rate to be constant. Here we allow the interest rate to change in time, \( r(t) \). Applying Black and Scholes’ no-arbitrage argument on Heston’s risk-free portfolio motivation for the evolution of the option price, we again obtain the partial differential equation (3.5) with \( r(t) \) rather than a constant \( r \) :

\[
\frac{\partial V}{\partial t} = -r(t)S \frac{\partial V}{\partial S} - \left\{ \kappa_0 [\theta - v] - \lambda v \right\} \frac{\partial V}{\partial v} - \frac{1}{2} \nu S^2 \frac{\partial^2 V}{\partial S^2} - \rho \sigma v S \frac{\partial^2 V}{\partial S \partial v} - \frac{1}{2} \sigma^2 v^2 \frac{\partial^2 V}{\partial v^2} \tag{3.30}
\]

For a given function \( r(t) \) this leads to a kernel \( P_V[S_T, v_T \mid S_0, v_0 \mid r(t)] \) so that the option price becomes

\[
C[r(t)] = \int_{-\infty}^{+\infty} dS_T dv_T \max [S_T - K, 0] \ e^{-\int_0^T r(t) dt} P_V[S_T, v_T \mid S_0, v_0 \mid r(t)]. \tag{3.31}
\]

Note that the option price is now a functional of the given time evolution of the interest rate \( r(t) \). As in the previous section, it is convenient to introduce new integration variables

\[
y(t) = \ln \left( \frac{S}{S_0} \right) - \frac{\rho}{\sigma} \left[ z^2(t) - \kappa t \right], \tag{3.32}
\]

\[
z(t) = \sqrt{v(t)}. \tag{3.33}
\]

In the path-integral treatment, the kernel can be written as a sum over all possible realizations of \( y(t) \) and \( z(t) \), weighed by the action functional of the system:

\[
C[r(t)] = \int_{-\infty}^{+\infty} dx_T dv_T \max [e^{x_T} - K, 0] \ e^{-\int_0^T r(t) dt} \times \int Dz \ Dz \exp \left( -\int_0^T \{ L_Q[y(t), z(t), r(t)] + L_{CIR}[z(t)] \} dt \right), \tag{3.34}
\]

where \( L_Q \) is the quadratic Lagrangian (3.18)

\[
L_Q(y(t), z(t)) = \frac{1}{2z^2 (1 - \rho^2)} \left[ y(t) - r(t) - \left( \frac{\rho}{\sigma} \kappa - \frac{1}{2} \right) z^2(t) \right]^2, \tag{3.35}
\]

and \( L_{CIR} \) is the CIR Lagrangian. Of course, we cannot know what particular realization of the interest rate \( r(t) \) will appear in the future. We assume the interest rate to follow a CIR process which is uncorrelated from the other two stochastic processes,

\[
\frac{dr}{\sqrt{r}} = \kappa_r (\theta_r - r) \ dt + \sigma_r \sqrt{r} dw_3. \tag{3.36}
\]
The value for the option price then needs to be averaged over the realization of \( r(t) \) in this CIR process. Where the calculation of the expectation value of such a functional might become cumbersome with conventional probabilistic techniques, it can be evaluated very elegantly with the Feynman-Kac formula:

\[
C = \langle C[r(t)] \rangle = \int DR \ C[r(t)] \exp \left( - \int_0^T \mathcal{L}_{CIR}[r(t)] dt \right),
\]

(3.37)

where \( \mathcal{L}_{CIR} \) is the Lagrangian for the CIR process corresponding to the interest rate. The final result can be expressed with a modified propagator \( P(S_T, v_T, r_T \mid S_0, v_0, r_0) \) as

\[
C = \int_0^\infty dS_T \int_0^\infty dv_T \int_0^\infty dr_T \max [S_T - K, 0] \ P(S_T, v_T, r_T \mid S_0, v_0, r_0),
\]

(3.38)

with

\[
P(S_T, v_T, r_T \mid S_0, v_0, r_0) = \int Dz DR D\vartheta \ e^{-\int_0^T r(t) dt} \times \exp \left( - \int_0^T \{ \mathcal{L}_Q[y(t), z(t), r(t)] + \mathcal{L}_{CIR}[z(t)] + \mathcal{L}_{CIRr}[r(t)] \} dt \right).
\]

(3.39)

The stochastic interest rate makes the vanilla price dependent on the specific path followed by the interest rate. This part of the payoff has been taken into the calculation of the propagator, where it is analytically tractable, and no longer appears explicitly in expression (3.38) for the option price. Herein lies the strength of the path-integral approach, to price path-dependent options. With a stochastic interest rate the European vanilla option becomes dependent on the entire path of the interest rate and is still solved in a very straightforward way.

A useful substitution to perform the functional integrations is

\[
\vartheta_1(t) = \sqrt{r(t)},
\]

\[
\vartheta_2(t) = y(t) - \int_0^t r(t') dt'.
\]

(3.40)

As was the case for the Lagrangian corresponding to the volatility, the Lagrangian corresponding to the interest rate process will also be formally equivalent to the Lagrangian corresponding to a radial harmonic oscillator; furthermore the addition of another term quadratic in \( \vartheta_1 \) stemming from the discount factor doesn’t spoil the correspondence. The result reads as follows:

\[
C = \frac{1}{2} \left[ S_0 - K \exp \left( \frac{\kappa_r}{\sigma_r^2} a_r + \Upsilon_r(0) \right) \right]
\]

\[
+ i \int_{-\infty}^1 \left\{ K \exp \left[ \Upsilon_r(0) + \frac{\kappa_r}{\sigma_r^2} a_r \right] - S_0 + \exp \left[ i \left( \frac{\rho}{\sigma} a + \theta_r \right) l + \frac{\kappa}{\sigma^2} a + \frac{\kappa_r}{\sigma_r^2} a_r \right] \right\} \frac{dl}{2\pi},
\]

(3.41)
To make the equation tractable, we introduced the following notations

\[
a_r = r_0 + \kappa_r \theta_r T,
\]

\[
\nu_r = \sigma_r \sqrt{\frac{\kappa_r^2}{\sigma_r^2} + 2il},
\]

\[
\omega_r (l) = \frac{\sigma_r}{2} \sqrt{\frac{\kappa_r^2}{\sigma_r^2} + 2(i l + 1)},
\]

\[
M_r = \left[ \cosh (\nu_r T) + \frac{\kappa_r}{2\nu_r} \sinh (\nu_r T) \right]^{-1},
\]

\[
\Theta_r = \frac{2 \nu_r r_0}{\sigma_r^2 \sinh (\nu_r T)} \left[ M_r - \cosh (\nu_r T) \right] + 2 \frac{\kappa_r \theta_r}{\sigma_r^2} \log M_r,
\]

\[
\Upsilon_r (l) = \frac{2 \omega_r (l) r_0}{\sigma_r^2 \sinh [\omega_r (l) T]} \left\{ N_r (l) - \cosh [\omega_r (l) T] \right\} + 2 \frac{\kappa_r \theta_r}{\sigma_r^2} \log N_r (l).
\]

These notations reflect the extension to the case of stochastic interest rate (symbols with subscript \( r \)) of the corresponding quantities in the Heston model (equations (3.29a)-(3.29f)). Notice the resemblance to formula (3.28). Formula (3.41) still contains just one numerical integration with an integrand composed out of elementary functions. To the best of our knowledge, only approximate analytical formulae are available when both the volatility and interest rate are stochastic [226]. Because of the lack of alternative exact analytical expressions, we have checked the validity of our formula (3.41) against numerical Monte Carlo simulations. Our Monte Carlo method is outlined below.

First notice that substitutions (3.40) transform the \( x \)-variable into a variable \( \tilde{x} \), independent of the interest rate by subtracting the time averaged interest rate \( \bar{r} \): \( \tilde{x} = x - \bar{r} \). This results in the same equation as in the constant interest rate situation, Eq. (3.13). Also the discount factor only contains \( \bar{r} \). This means that the knowledge of the probability distribution \( \bar{r} \) is sufficient to calculate the price by means of the formula (3.28) derived in the constant interest rate setting. So the Monte Carlo scheme used is the following: first values for \( \bar{r} \) are simulated and used to calculate the option price for these values, next the price is averaged over all the simulations. A value for \( \bar{r} \) is simulated as follows: time is discretized in little time steps \( \Delta \), then we sample a path for \( r \) and integrate along this path. To calculate the probability distribution for \( \bar{r} \), we used the result that the stochastic time increment of a CIR variable over a small time step \( \Delta t \) follows a non-central \( \chi^2 \) distribution [227]. The probability distribution of the average interest rate \( \bar{r} \) is then simulated by generating many \( r \)-paths in discretized time. As shown in Fig. 3.1, the agreement between the analytical (thick full line) and numerical option prices is excellent.

In this section the option propagation approach was followed from the beginning. In this setting it is necessary to make a choice between the two approaches from the start because in the asset propagation approach one would actually have to introduce a stochastic process for the drift \( \mu_0 \) instead of for the interest rate. That these two should follow the same stochastic process is not clear. Since the option propagation approach is the most common one anyway we followed this approach. If one does want to introduce a stochastic process for the drift \( \mu_0 \) this poses no
problem and the derivation of an option price in this setting would be completely similar.

3.2.2 Results and discussion.

In the current treatment, we have two layers of generalization as compared to the Black-Scholes result see Fig. 3.2.2. First, the volatility appearing in the Black-Scholes process is stochastic – this leads to the Heston model. Second, the interest rate of the Black-Scholes model is also stochastic – leading to our present results. In this paragraph, we argue that both improvements can have an equally important effect on the option price.

Figure 3.1: This figure shows the result of different pricing formulas from which the Black-Scholes result (with interest rate $r = \theta$, $\sigma = \sqrt{\theta}$) has been subtracted (these prices are shown in Fig. 3.2.2). Since we are not considering a specific asset, the option price could be stated in any currency, therefore the deviation is given in arbitrary units. The thick (red) curve shows our analytical results for the model with both stochastic interest rate $r$ and stochastic volatility. The crosses represent results from a Monte Carlo simulation of our model, confirming the analytical formula. The blue dotted curve and the blue dash-dotted curve show the results for the Heston model with constant interest rate $r = r(0)$ and $r = \theta$, respectively. The dashed curve shows the results for a Black-Scholes model with $r = r(0)$. The following parameter values were used for all three panels: $\kappa = 1$, $\sigma = 0.2$, $\theta = 0.04$, $v_0 = 0.04$, $T = 1$, $S_0 = 100$, $\kappa_r = 1.8$; $\sigma_r = 0.1$; $\theta_r = 0.03$; $r_0 = 0.035$. The correlation coefficient is for panel (a) $\rho = -0.5$, (b) $\rho = 0$ and (c) $\rho = 0.5$.

This is illustrated in Fig. 3.1, where the different approaches are compared. Let’s start with the most complete model, where both interest rate and volatility are stochastic. The resulting option price, Eq. (3.41), for this model is shown as a thick red curve. The result from the closed-form expression agrees well with the Monte Carlo simulation, shown as crosses.

Now we strip off one layer of complexity, and fix the interest rate $r$ – it is no longer a stochastic variable. Then we obtain the Heston model as an ‘approximation’ to a stochastic interest rate world. The question poses itself of which fixed interest rate to use, if we want to make the comparison. Two choices are shown in Fig. 3.1: $r = r(0)$ and $r = \theta$. The former choice (dotted blue curves) sets the Heston
interest rate equal to the interest rate at time 0, whereas the latter choice (dash-dotted curves) sets the Heston interest rate equal to the mean reversion level \( \theta_r \). For the parameter values used in Fig. 3.1, the most complete result lies between the two Heston ‘approximations’, but this is not necessarily so. Fig. 3.3 shows that for some choices of other (realistic) parameters, the full result can lie outside both Heston approximations. Nevertheless, as \( \kappa \) becomes very large, the stochastic interest rate will be drawn very tightly to the mean reversion rate \( \theta_r \), and one expects the full result to be near the Heston approximation with \( r = \theta_r \). When \( \kappa \) is very small, the stochastic interest rate will not be drawn quickly towards \( \theta_r \) so that when also \( \sigma_r \) is small, the full results will be near the Heston approximation with \( r = r(0) \).

Next, we strip off the second layer of approximation, and also fix the volatility. This results in the familiar Black-Scholes model as the crudest approximation to our system. Now a second choice has to be made: which value of the volatility to use. Here, we take the stochastic volatility at time zero to be equal to the mean reversion level of the volatility CIR process, so that the ambiguity of choice is avoided. The choice for what interest rate to use, however, remains. In Fig. 3.1, we show the Black-Scholes results with \( r = r(0) \) (dashed line) and \( r = \theta_r \) (full line). We have plotted all the results relative to the Black-Scholes result with \( r = \theta_r \) to emphasize the differences rather than the absolute magnitude of the prices (for this reason, the \( r = \theta_r \) Black-Scholes result is the baseline of the plots). The difference between the three panels of Fig. 3.1 is the value of the correlation between asset price and volatility.

From Figs. 3.1 and 3.3, it is clear that both levels of approximation (keeping the volatility constant and keeping the interest rate constant) have an equally large effect on the option price. Even within the Heston framework, the choice of what value to use for the interest rate is seen to influence the price considerably. Choosing a different interest rate, or keeping the interest rate as a stochastic variable, leads to a price correction that is as large as the price correction obtained by going from the Black-Scholes to the Heston model. This result emphasizes the importance of a correct treatment of the interest rate in pricing models (This also depends strongly on the length of the lifetime of the option).

Finally we must remark that the price differences when working within the stan-
Figure 3.3: As in Fig. 1, the result of different pricing formulas from which the Black-Scholes result (with interest rate \( r = \theta_r \), thin black line) has been subtracted, is shown. The following parameter values were used: \( \kappa = 1 \), \( \sigma = 0.2 \), \( \theta = 0.04 \), \( v_0 = 0.04 \), \( T = 1 \), \( S_0 = 100 \), \( \kappa_r = 0.5 \), \( \sigma_r = 0.3 \), \( \theta_r = 0.03 \), \( r_0 = 0.035 \), \( \rho = 0 \).

Standard Heston model or within the extended one can be influenced by the calibration method. For Figures. 3.1 and 3.3 we used the same parameters for the volatility process both in the standard model and in the extended one. Parameter values for the interest rate process are calibrated separately. Literature shows that the parameter values for the volatility process (see for example [217] and [10]) and the interest rate process (see for example [223] and [221]) can attain values in a broad range containing the values we chose to produce Fig. 3.1 and Fig. 3.3. However if the parameter values obtained for the interest rate process are used in formula (3.41) to calibrate the remaining parameter values for the volatility process one might get different results. We can not exclude that this calibration approach would lead to smaller price differences between the two approaches. However such a calibration is a research area on its own and is outside the scope of this work.
Chapter 4

Geometric Asian options in the Black-Scholes model*

In this chapter we discuss the pricing of the geometrically averaged Asian option in the Black-Scholes model. In section 4.1 the time evolution of the joint density of the asset price and its average is derived with the path integral framework in a similar fashion as Linetsky [159] derives it for the weighted Asian option. In section 4.2 we present an alternative derivation of this propagator using a stochastic calculus approach. This propagator now allows us to price both the average price and average strike Asian option. For both types of options this results in a pricing formula which is of the same form as the Black-Scholes formula for the plain vanilla option. Our result for the option price of an average price Asian option confirms the result obtained by Linetsky [159] and Lipton [210]. For the average strike option no formula of this simplicity existed, as far as we know. Our derivation and analysis of this formula is presented in section 4.3, where our result is checked with a Monte Carlo simulation.

4.1 Partitioning the set of all paths

The path integral propagator is used in financial science to track the probability distribution of the logreturn \( x_t = \log(S_t/S_0) \) at time \( t \), where \( S_0 \) is the initial value of the underlying asset. This propagator is calculated as a weighted sum over all paths from the initial value \( x_0 = 0 \) at time \( t = 0 \) to a final value \( x_T = \log(S_T/S_0) \) at time \( t = T \):

\[
P(x_T|0) = \int \mathcal{D}x \exp \left( - \int_0^T \mathcal{L}_{BS} \left[ x(t) \right] dt \right)
\]  

(4.1)

The weight of a path, in the Black-Scholes model, is determined by the Lagrangian

\[
\mathcal{L}_{BS} \left[ x(t) \right] = \frac{1}{2\sigma^2} \left[ \dot{x} - \left( \mu - \frac{\sigma^2}{2} \right) \right]^2
\]  

(4.2)

---

*This chapter is based on Ref. [228], which is joint work with Jeroen Devreese and Jacques Tempère. Jeroen is the first author of this paper and the main contributor. I only included those parts of the article for which I have the feeling I made important contributions. Furthermore the work is inspired by discussions with Sven Foulon and Karel in ’t Hout.
where $\mu$ is the drift and $\sigma$ is the volatility appearing in the Wiener process for the logreturn $[224]$.

For Asian options, the payoff is a function of the average value of the asset. Therefore we introduce $\bar{x}_T = \log (\bar{S}_T/S_0)$ as the logreturn corresponding to the average asset price at maturity $T$. When $\bar{S}_T$ is the geometric average of the asset price, then $\bar{x}_T$ is the arithmetic average of $x$.

$$\bar{x}_T = \frac{1}{T} \int_0^T x(t) dt.$$  \hspace{1cm} (4.3)

$$\bar{S}_T = S_0 \exp (\bar{x}_T),$$  \hspace{1cm} (4.4)

$$= S_0 \exp \left( \frac{1}{T} \int_0^T x(t) dt \right),$$  \hspace{1cm} (4.5)

$$= \exp \left( \frac{1}{T} \int_0^T \ln [S(t)] dt \right).$$  \hspace{1cm} (4.6)

The key step to treat Asian options within the path integral framework is to partition the set of all paths into subsets of paths, where each path in a given subset has the same $\bar{x}_T$. This technique was developed in [229], for the purpose of reducing many complicated path integrals to simple ordinary integrals, and has led to the powerful calculus of variational perturbation theory [202]. Later Linetsky [159] observed that this technique could be used in the context of Asian options. The joint propagator $P(x_T, \bar{x}_T|0,0)$ becomes:

$$P(x_T, \bar{x}_T|0,0) = \int Dx \delta (\bar{x}_T - \frac{1}{T} \int_0^T x(t) dt) \exp \left( - \int_0^T \mathcal{L}_{BS} [x(t)] dt \right) \hspace{1cm} (4.7)$$

The delta function in the sum $\int Dx$ over all paths picks out precisely all the paths that will have the same payoff for an Asian option.

The calculation of $P(x_T, \bar{x}_T|0,0)$ is straightforward; when the delta function is rewritten as a Fourier integral,

$$P(x_T, \bar{x}_T|0,0) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik\bar{x}_T} \int Dx \exp \left( - \int_0^T \left( \mathcal{L}_{BS} [x(t)] + \frac{1}{T} ik x(t) \right) dt \right), \hspace{1cm} (4.8)$$

the resulting Lagrangian is that of a free particle in a constant force field in 1D. The resulting integration over paths is found by standard procedures [201]:

$$P(x_T, \bar{x}_T|0,0) = \frac{\sqrt{3}}{\pi \sigma^2 T} \exp \left\{ - \frac{1}{2\sigma^2 T} \frac{T}{2} \left[ x_T - \left( \mu - \frac{\sigma^2}{2} \right) T \right]^2 - \frac{6}{\sigma^2 T} \left( \bar{x}_T - \frac{x_T}{T} \right)^2 \right\},$$  \hspace{1cm} (4.9)

and corresponds to the result found by Refs. [159, 202].

### 4.2 Link with stochastic calculus

The calculation of $P(x_T, \bar{x}_T|0,0)$ here is similar to the derivation presented in [230] where this joint propagator is calculated for the Vasicek model. The main point
is that in a Gaussian model the joint distribution of the couple \( \{ x_T, \bar{x}_T \} \) has to be Gaussian too. As a consequence this joint distribution is fully characterized by the expectation values and the variances of \( x_T \) and \( \bar{x}_T \) and by the correlation between these two processes. The expectation value of \( \bar{x}_T(T) \) can easily be found:

\[
\mu_{\bar{x}} = \mathbb{E} \left[ \frac{1}{T} \int_0^T x(t) \, dt \right] = \frac{1}{T} \int_0^T \mathbb{E} [x(t)] \, dt = \frac{\left( \mu - \frac{\sigma^2}{2} \right) T}{2T} = \frac{\mu - \frac{\sigma^2}{2}}{2} T,
\]

as well as its variance:

\[
\frac{1}{T^2} \text{Var} \left[ \int_0^T x(t) \, dt \right] = \frac{2}{T^2} \int_0^T \int_0^t \text{cov} (x(t), x(u)) \, dudt = \frac{2\sigma^2}{T^2} \int_0^T \int_0^t u \, dudt = \frac{\sigma^2 T}{3}, \]

and its correlation:

\[
cov \left( x(T), \frac{1}{T} \int_0^T x(u) \, du \right) = \frac{1}{T} \int_0^T \int_0^t \text{cov} (x(t), x(u)) \, dudt = \frac{1}{T} \sigma^2 \int_0^T u \, du = \frac{\sigma^2 T}{2},
\]

\[
corr (x, \bar{x}) = \frac{\text{cov} (x, \bar{x})}{\sigma_x \sigma} = \frac{\frac{\sigma^2 T}{2}}{\sigma \sqrt{\frac{T}{3}} \sigma} = \frac{\sqrt{3}}{2}.
\]

The density function of such a Gaussian process is then known to be

\[
P(x_T, \bar{x}_T, |0,0) = \frac{\sqrt{3}}{\pi \sigma^2 T} \exp \left( -\frac{2}{\sigma^2 T} \left\{ \left[ x_T - \left( \mu - \frac{\sigma^2}{2} \right) T \right]^2 + 3 \left[ \bar{x}_T - \left( \mu - \frac{\sigma^2}{2} \right) \frac{T}{2} \right] - 3 \left[ x_T - \left( \mu - \frac{\sigma^2}{2} \right) T \right] \left[ \bar{x}_T - \left( \mu - \frac{\sigma^2}{2} \right) \frac{T}{2} \right] \right\} \right).
\]

This agrees with Eq. (4.9) for \( P(x_T, \bar{x}_T|0,0) \).

### 4.3 Pricing of an average strike geometric Asian option

If the payoff at time \( T \) of an Asian option is written as \( V_T^{\text{Asian}}(x_T, \bar{x}_T) \), then the expected payoff is

\[
\mathbb{E} \left[ V_T^{\text{Asian}}(x_T, \bar{x}_T) \right] = \int_{-\infty}^{\infty} dx_T \int_{-\infty}^{\infty} d\bar{x}_T \; V_T^{\text{Asian}}(x_T, \bar{x}_T) P(x_T, \bar{x}_T|0,0) \quad (4.10)
\]

The price of the option, \( V_0^{\text{Asian}} \) is the discounted expected payoff,

\[
V_0^{\text{Asian}} = e^{-rT} \mathbb{E} \left[ V_T^{\text{Asian}}(x_T, \bar{x}_T) \right] \quad (4.11)
\]

where \( r \) is the discount (risk-free) interest rate. Using expression (4.10) the price of any option which is dependent on the average of the underlying asset during the
lifetime of the option can be calculated. We will now derive the price of an average strike geometric Asian call option explicitly. In order to do this, expression (4.10) has to be evaluated using the payoff:

\[
V_{\text{Asian}}^{\text{Asian}}(x_T, \bar{x}_T) = \max(S_T - \bar{S}_T, 0) = S_0 \max(e^{x_T} - e^{\bar{x}_T}, 0)
\] (4.12)

Substituting (4.12) in (4.10) yields

\[
V^{\text{Asian}}_0 = S_0 e^{-rT} \int_{-\infty}^{\infty} d\bar{x}_T \int_{-\infty}^{\infty} dx_T \ (e^{x_T} - e^{\bar{x}_T}) P(x_T, \bar{x}_T|0, 0)
\] (4.13)

where the lower boundary of the \(x_T\) integration now depends on \(\bar{x}_T\). When considering an average price call, the payoff (for a call option) is \(\max(\bar{S}_T - K, 0)\) leading to a constant lower boundary \(\log(\frac{K}{S_0})\) for the \(\bar{x}_T\) integration, and the integrals are easily evaluated. In the present case however, the integration boundary is more complicated and it is more convenient to express this boundary through a Heaviside function, written in its integral representation:

\[
\theta(x_T - \bar{x}_T) = \int_{-\infty}^{+\infty} d\tau \ e^{i(x_T - \bar{x}_T)\tau} / (\tau - i\varepsilon).
\]

We obtain:

\[
V^{\text{Asian}}_0 = S_0 e^{-rT} \frac{\sqrt{3}}{2\pi i} \int_{-\infty}^{+\infty} d\bar{x}_T \int_{-\infty}^{+\infty} dx_T \int_{-\infty}^{+\infty} d\tau \ e^{i(x_T - \bar{x}_T)\tau} / (\tau - i\varepsilon)
\]

\[
\times (e^{x_T} - e^{\bar{x}_T}) P(x_T, \bar{x}_T|0, 0)
\] (4.14)

Now the two original integrals have been reduced to Gaussians at the cost of inserting a complex term in de exponential. Expression (4.14) can be split in two terms denoted \(I_1\) and \(I_2\), where

\[
I_1 = S_0 e^{-rT} \frac{\sqrt{3}}{\pi \sigma^2 T} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} d\tau \ \frac{1}{\tau - i\varepsilon} \int_{-\infty}^{+\infty} d\bar{x}_T \int_{-\infty}^{+\infty} dx_T \ e^{i(x_T - \bar{x}_T)\tau} / (\tau - i\varepsilon)
\]

\[
\times \exp \left\{ -\frac{1}{2\sigma^2 T} \left[ x_T - \left( \mu - \frac{\sigma^2}{2} \right) T \right]^2 - \frac{6}{\sigma^2 T} (\bar{x}_T - \frac{x_T}{2})^2 + i(x_T - \bar{x}_T)\tau + x_T \right\}
\] (4.15)

and \(I_2\) has the same form, except with \(\bar{x}_T\) instead of \(x_T\) in the last term of the argument of the exponent. As a first step, the Gaussian integrals over \(x_T\) and \(\bar{x}_T\) are calculated, yielding

\[
I_1 = S_0 e^{-(r - \mu)T} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \exp \left[ -\frac{\sigma^2 T}{6} \tau^2 + \left( \mu + \frac{\sigma^2}{2} \right) \frac{\sigma^2 \tau}{2} \right] d\tau
\]

\[
= S_0 e^{-(r - \mu)T} \sqrt{\frac{3}{2\pi \sigma^2 T}} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} d\tau \ e^{i\varepsilon \tau} \exp \left[ -\frac{3}{2\sigma^2 T} \left( l - \left( \mu + \frac{\sigma^2}{2} \right) \frac{\sigma^2}{2} \right)^2 \right].
\] (4.16)
Recognizing in the $\tau$ integral the Fourier transform of the Heaviside function the next expression follows

$$I_1 = S_0 e^{-(r-\mu)T} \sqrt{\frac{3}{2\pi\sigma^2T}} \int_0^{+\infty} \exp \left[ -\frac{3}{2\sigma^2T} \left( l - \frac{\left( \mu + \frac{\sigma^2}{2} \right)^2}{2} \right) \right] dl$$

$$= S_0 e^{-(r-\mu)T} N \left( \frac{\left( \mu + \frac{\sigma^2}{2} \right)}{2\sqrt{3\sigma^2T}} \right),$$

with $N$ the normal cumulative distribution function. The second term, $I_2$, is evaluated similarly, leading to the following price for the Asian option

$$V_0^{Asian} = S_0 e^{-rT} \left( e^{\mu T} N \left( \sqrt{\frac{3T}{4\sigma^2}} \left( \mu + \frac{\sigma^2}{2} \right) \right) - e^{(\mu - \frac{\sigma^2}{6}) T} N \left( \sqrt{\frac{3T}{4\sigma^2}} \left( \mu - \frac{\sigma^2}{6} \right) \right) \right).$$

(4.17)

### 4.4 Discussion

Expression (4.17) is the analytic pricing formula for an average strike geometric Asian call option, obtained in the present work with the path integral formalism. To the best of our knowledge, no pricing formula of this simplicity exists. To check this formula, we compared its results to those of a Monte-Carlo simulation. The Monte Carlo scheme used is as follows [230]: first, the evolution of the logreturn is simulated for a large number of paths. This evolution is governed by a discrete time geometric Brownian motion for a number of time steps. Using the value for the logreturn at each time step, the average logreturn can be calculated for every path. Subsequently the payoff per path can be obtained, which is then used to calculate the option price by averaging over all payoffs per path and discounting back in time. The analytical result and the Monte Carlo simulation agree to within a relative error of 0.3% when 500000 samples and 100 time steps are used. We also obtained the result for an average price Asian option; in contrast to the new result for the average strike option this could be compared to existing formulas [159], and was found to be the same. Fig. 4.1 shows the price of the average strike and average price Asian call option together with the price of the vanilla call option. Since the average asset price is less likely to attain high values than the asset price at maturity, it follows that the price of the average price Asian option is lower than the vanilla call option price.
Figure 4.1: Here the vanilla call option price is presented together with the price of an average price Asian call option and an average strike Asian option. Parameters are: $r = 0.03$, $S_0 = 100$, $\sigma = 0.2$. 

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Chapter 5

Generalized pricing formulas for stochastic volatility jump diffusion models applied to the exponential Vašiček model*

In this chapter we will present a method that makes it possible to extend the Fourier space propagator of a general SV model to the Fourier space propagator of that SV model where an arbitrary jump process has been added to the asset price dynamics. Thereby we contribute to the existing work on Fourier transform methods applied to option pricing. For example in [82] jump diffusions are treated and prices for some exotic options are obtained. In [232] the Heston model is extended with a jump process for the asset price. In [188] the Heston model is extended with arbitrary jump processes in both the asset price and the volatility process.

As an application, we investigate a model where we assume that the stochastic volatility follows an exponential Vašiček model [5, 233]. To the best of our knowledge, for this model no closed form formulas for the propagator or the vanilla option price exist yet. Making use of path integral methods [194, 202, 234] we derive approximative closed form formulas for the propagator and for vanilla option prices for this model. Using Monte Carlo (MC) simulations we specify parameter ranges for which the approximation is valid. Using the above mentioned method we extend the propagator of this model to the propagator of this model including jumps in the asset price which leads also to closed form pricing formulas in this extended model. Also these last results are checked with MC simulations.

This chapter is organized as follows. In section 5.1 we present the method for extending the propagator of a general SV model to the propagator of that model with jumps in the asset price. In section 5.2, we present an approximative propagator for jump diffusion models where the volatility is assumed to follow an exponential Vašiček model. Section 5.3 is devoted to European vanilla option pricing, as well as

*This chapter is based on Ref. [231], which is joint work with Ling Zhi Liang and Jacques Tempère. Ling Zhi is the first author of this paper and the main contributor. In contrast with the previous chapter it is impossible now to specify and omit parts to which I didn’t contribute. Furthermore we acknowledge discussions with Sven Foulon and Karel in ’t Hout.
comparisons with MC simulations. In this section we also give parameter ranges for the approximation made in the exponential Vašíček model to be valid.

5.1 General propagator formulas

5.1.1 Arbitrary SV models

We assume that the asset price process $S(t)$ follows the Black-Scholes stochastic differential equation (SDE):

$$dS = rSdt + \sigma SdW_1,$$

(5.1)

in which $r$ is the constant interest rate and the volatility $\sigma(t)$ is behaving stochastically over time, following an arbitrary Itô stochastic process:

$$d\sigma = A(t, \sigma)dt + B(t, \sigma)dW_2.$$  

(5.2)

Here and in the rest of this chapter $W_j = \{W_j(t), t \geq 0\}$ $(j = 1, 2)$ are two correlated Wiener processes such that $\text{Cov}[dW_1(t), dW_2(t)] = \rho dt$.

Eq.(5.1) is commonly expressed as a function of the logreturn $x(t) = \ln S(t)$, which leads to a new SDE:

$$dx = \left(r - \frac{1}{2} \sigma^2\right) dt + \sigma dW_1.$$  

(5.3)

To deal with the pricing problem, we need to solve for the propagator of the joint dynamics of $x(t)$ and $\sigma(t)$. The propagator, denoted by $\mathcal{P}(x_T, \sigma_T|x_0, \sigma_0)$, describes the probability that $x$ has the value $x_T$ and $\sigma$ has the value $\sigma_T$ at a later time $T$ given the initial values $x_0$ and $\sigma_0$ respectively at time 0. It satisfies the following Kolmogorov forward equation:

$$\frac{\partial \mathcal{P}}{\partial T} = \frac{\partial}{\partial x_T} \left[-(r - \frac{1}{2} \sigma_T^2)\mathcal{P}\right] + \frac{1}{2} \frac{\partial^2}{\partial x_T^2} \left[\sigma_T^2 \mathcal{P}\right] + \frac{\partial}{\partial \sigma_T} \left[-A(T, \sigma_T)\mathcal{P}\right] + \frac{1}{2} \frac{\partial^2}{\partial \sigma_T^2} \left[B^2(T, \sigma_T)\mathcal{P}\right] + \rho \frac{\partial}{\partial x_T \partial \sigma_T} \left[\sigma_T B(T, \sigma_T)\mathcal{P}\right].$$  

(5.4)

5.1.2 SV jump diffusion models

A general SV jump diffusion model is obtained by adding an arbitrary jump process into the asset price process (see for instance [144]). That is, equation (5.1) becomes

$$dS = \mu Sdt + \sigma(t) SdW_1(t) + (e^J - 1) SdN,$$

(5.5)

where $N = \{N_t, t \geq 0\}$ is an independent Poisson process with intensity parameter $\lambda > 0$, i.e. $\mathbb{E}[N(t)] = \lambda t$. The random variable $J$ with probability density $\varpi(J)$ describes the magnitude of the jump when it occurs.

Here the risk-neutral drift $\mu = r - \lambda m^j$ is no longer the constant interest rate $r$, rather it is adjusted by a compensator term $\lambda m^j$, with $m^j$ the expectation value of $e^J - 1$:

$$m^j = \mathbb{E}[e^J - 1] = \int_{-\infty}^{+\infty} (e^J - 1) \varpi(J) dJ.$$  

(5.6)
so that the asset price process constitutes a martingale under the risk neutral measure. And the logreturn $x(t)$ follows a new SDE:

$$dx = \left( r - \lambda m^j - \frac{1}{2} \sigma^2(t) \right) dt + \sigma(t) dW_t + J dN.$$  (5.7)

Given the same arbitrary SV process (5.2), the new propagator of this model, denoted by $P_J(x_T, \sigma_T, T|x_0, \sigma_0, 0)$, satisfies the new Kolmogorov forward equation (see for instance [235])

$$\frac{\partial P_J}{\partial T} = \frac{\partial}{\partial x_T} \left[ - \left( r - \lambda m^j - \frac{1}{2} \sigma^2_T \right) P_J \right] + \frac{1}{2} \frac{\partial^2}{\partial x_T^2} \left[ \sigma^2_T P_J \right] + \frac{\partial}{\partial \sigma_T} \left[ - A(T, \sigma_T) P_J \right]
+ \frac{1}{2} \frac{\partial^2}{\partial \sigma_T^2} \left[ \sigma^2_T P_J \right] + \rho \frac{\partial}{\partial \sigma_T} \left[ \sigma_T B(T, \sigma_T) P_J \right]
+ \lambda \int_{-\infty}^{+\infty} \left[ P_J(x_T - J) - P_J(x_T) \right] \varpi(J) dJ.$$  (5.8)

If we write the propagator of the arbitrary SV model as a Fourier integral

$$P(x_T, \sigma_T|x_0, \sigma_0) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi} e^{ip(x_T-x_0)} F(\sigma_T, \sigma_0, r, p, T),$$  (5.9)

then the propagator of arbitrary SV jump diffusion models can be written as

$$P_J(x_T, \sigma_T|x_0, \sigma_0) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi} e^{ip(x_T-x_0)} F(\sigma_T, \sigma_0, r, p, T) e^{U(p,T)},$$  (5.10)

where

$$U(p, T) = \lambda T \int_{-\infty}^{+\infty} \left[ e^{-ip\lambda} - 1 + ip \left( e^{\lambda} - 1 \right) \right] \varpi(J) dJ.$$  (5.11)

The proof of this statement is given in the Appendix 5.A. Note the relation between propagators (5.9) and (5.10). The only difference between them is the factor $e^{U(p,T)}$.

If this is applied to the propagator of the Heston model (equation (3.25)), the propagator of the Heston model with jumps is obtained. This propagator is similar as the one derived in Ref. [188]. Furthermore the above described method can be combined with the method described in chapter 3 for finding the propagator of a model including both SV and stochastic interest rate. In particular extending the result of chapter 3 for the Heston model with stochastic interest rate to include jumps again only involves multiplying the propagator with $e^{U(p,T)}$ as in (5.10). In the next section, as an example of the method of this section the volatility of the asset price will be assumed to follow an exponential Vašiček model.

### 5.2 Exponential Vašiček SV model with price jumps

The Heston model assumes that the squared volatility follows a CIR process which has a gamma distribution as stationary distribution. This assumption should be
compared with market data. Attempts to reconstruct the stationary probability
distribution of volatility from the time series data (among others, see Refs. [5, 36,
233]) generally agree that the central part of the stationary volatility distribution is
better described by a lognormal distribution.

Ref. [104] finds that due to the different structure in path-behavior between
different models, the resulting exotic option prices can vary significantly. So an
investigation into an alternative model which fits market data better is meaningful.
Furthermore the model will serve here both to demonstrate the use of path integral
methods in finance and to illustrate the method of section 5.1.

When $\sigma(t)$ is assumed to be an exponential Vašíček (EV) process (used for
example by Chesney and Scott [80]), this results in the following two SDEs

$$dS = rS \, dt + \sigma S \, dW_1,$$

$$d\sigma = \sigma \left( \beta [\bar{a} - \ln \sigma] + \frac{1}{2} \gamma^2 \right) \, dt + \gamma \sigma dW_2.$$  

This model has a lognormal stationary volatility distribution. The propagator for
this model will be denoted by $P_{EV}$. In this model $\ln \sigma(t)$ is a mean reverting process,
with $\beta$ the spring constant of the force that attracts the logarithm of asset volatility
to its mean reversion level $\bar{a}$. Again $\gamma$ is the volatility of the asset volatility. As far
as we know, there is no closed form option pricing formula for this model. In this
section, we will give an approximation for the propagator of this model. In the next
section we will give an approximation for the vanilla option price and determine a
parameter range for which the approximation is good. The derivation starts with
the following substitutions:

$$y(t) = x(t) - \frac{\rho}{\gamma} e^{z(t)} - rt,$$

$$z(t) = \ln \sigma(t),$$

where $x(t)$ is defined as before. This leads to two uncorrelated equations:

$$dy = \left[ -\frac{1}{2} e^{2z} - \rho \left( \frac{\beta (\bar{a} - z)}{\gamma} + \frac{\gamma}{2} \right) e^z \right] \, dt + e^z \sqrt{1 - \rho^2} dB_1,$$

$$dz = \beta (\bar{a} - z) \, dt + \gamma dB_2,$$

where $B_1$ and $B_2$ are two uncorrelated Wiener processes. Since these equations
are uncorrelated, the propagator $P_{EV}(y_T, z_T | y_0, z_0)$ is given by the following path
integral

$$P_{EV}(y_T, z_T | y_0, z_0) = \int Dz \left( \int Dy e^{-\int_0^T \mathcal{L}_1[y,z] \, dt} \right) e^{-\int_0^T \mathcal{L}_2[z] \, dt},$$

where the Lagrangians are given by:

$$\mathcal{L}_1[y, z] = \frac{\dot{y} + \frac{1}{2} e^{2z} + \rho \left( \frac{\beta (\bar{a} - z)}{\gamma} + \frac{\gamma}{2} \right) e^z}{2(1 - \rho^2) e^{2z}},$$

$$\mathcal{L}_2[z] = \frac{\dot{z} - \beta (\bar{a} - z)^2}{2\gamma^2} - \frac{\beta}{2}.$$
The first step in the evaluation of (5.18) is the integration over all \( y \) paths. Because the action is quadratic in \( y \), this path integration can be done analytically and yields

\[
\mathcal{P}_{EV}(y_T, z_T | y_0, z_0) = \int \mathcal{D} z e^{-\int_0^T \mathcal{L}_2[z] dt} \frac{1}{\sqrt{2\pi(1 - \rho^2) \int_0^T e^{2z dt}}} 
\times e^{-\frac{[\psi_T - \psi_T^0 + \frac{\lambda}{2} \int_0^T (\tilde{a} \psi_T^0 + \frac{\tilde{a}}{2}) e^{\epsilon T} dt \nu r]}{2(1 - \rho^2) \int_0^T e^{2z dt}}}.
\]

(5.21)

Note that the probability to arrive in \((y_T, z_T)\) only depends on the average value of the volatility along the path \(z(t)\), in agreement with Ref. [80]. With the help of a Fourier transform, we rewrite the preceding expression as follows

\[
\mathcal{P}_{EV}(y_T, z_T | y_0, z_0) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi} e^{ip(y_T - y_0)} \int \mathcal{D} z e^{-\int_0^T \mathcal{L}_2[z] dt} 
\times e^{-\frac{(1 - \rho^2)p^2 - ip \psi_T - ip \psi_T^0 \left( \frac{\beta (\tilde{a} - \tilde{a})}{2} + \frac{\tilde{a}}{2} \right) e^{\epsilon T}}{2(1 - \rho^2) \int_0^T e^{2z dt}}}.
\]

(5.22)

If \(\zeta(t) = z(t) - \bar{a}\), then \(\zeta(t)\) is close to zero because \(z(t)\) is a mean reverting process with mean reversion level \(\bar{a}\). This motivates the approximation \(e^\zeta \approx 1 + \frac{1}{2} \zeta + \frac{1}{2} \zeta^2\). This type of approximation is akin to expanding the path integral around the saddle point up to second order in the fluctuations, as in the Nozieres-Schmitt-Rink formalism [237] extended to path-integration by Sa de Melo, Randeria and Engelbrecht [238] for superfluidity in quantum gases. Now we can work out the remaining path integral in (5.22)
We see that also the integral over the final value $\zeta_T$ can be done, yielding the marginal probability distribution:

$$P_{EV}(x_T|x_0, \zeta_0) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi} e^{ip(x_T-x_0-\tau_T)+B(\xi_0-1)}$$

$$\times e^{\frac{\beta \xi_0^2 - \omega (N + \frac{\gamma^2}{\omega^2})^2}{2\tau^2}} + \left[ \frac{\beta - \omega - A + B \gamma^2}{2\tau} + \frac{\gamma^2}{2\tau^2} \right] T$$

$$\times \frac{e^{-\tau^2(\beta - \omega - B \gamma^2)[1 - e^{-2\omega T}]}}{\sqrt{1 + \frac{1-e^{-2\omega T}}{2\omega} [\beta - \omega + B \gamma^2]}}.$$  \hspace{1cm} (5.28)

where

$$\Xi = \omega \left[ 2B \gamma^2 N + \omega(N - \frac{\gamma^2}{\omega^2})^2 - (\beta + B \gamma^2)N^2 \right]$$

$$+ (1 - e^{-2\omega T}) \left[ \frac{B^2 \gamma^4}{2} - B \gamma^4 M + \frac{(\beta + B \gamma^2)M^2}{2\omega^3} \right],$$  \hspace{1cm} (5.29)

$$N = \frac{\gamma^2 M}{\omega^2} - (\zeta_0 + \frac{\gamma^2 M}{\omega^2}) e^{-\omega T}.$$  \hspace{1cm} (5.30)

Figure 5.1: Propagator $P(x_T|x_0, \zeta_0)$ as a function of $x_T - x_0$. The full curves are our analytical results, while the symbols represent Monte Carlo simulations. $T = 0.25y$, $\rho = 0$ (crosses). $T = 1y$, $\rho = -0.5$ (circles). $T = 5y$, $\rho = 0.5$ (triangles). For the other parameters the following values are used for the three figures: $\beta = 5$, $\vec{a} = -1.6$, $\gamma = 0.5$, $r = 0.015$.

This approximative propagator was checked with MC simulations, as there are no closed-form solutions in literature to compare it to. Figure 5.1 shows the propagators as a function of $x_T - x_0$, i.e., $\ln S_T$. The full curves come from expression (5.28),
while the marked ones are MC simulation results, with time to maturity ranging from three months to five years, and correlation coefficients 0, $-0.5$ and $0.5$ respectively. Here and in the rest of the chapter we will set $\sigma_0$ equal to the long time average of the volatility:

$$\sigma_0 = \lim_{t \to \infty} \mathbb{E} [\sigma(t)] = \exp \left\{ \bar{a} + \frac{\gamma^2}{4\beta} \right\}, \quad (5.31)$$

which seems a reasonable choice. For these MC simulations 5,000,000 sample paths are used.

It is seen that our analytical results fit the MC simulations quite well. Actually, using the parameters of Fig. 5.1, and putting expression (5.28) for those three cases into the left hand side of the Kolmogorov backward equation:

$$- \frac{\partial P}{\partial T} + \left[ r - \frac{1}{2} e^{2(\zeta_0 + \bar{a})} \right] \frac{\partial P}{\partial x_0} + \frac{1}{2} e^{2(\zeta_0 + \bar{a})} \frac{\partial^2 P}{\partial x_0^2} - \beta \zeta_0 \frac{\partial P}{\partial \xi_0} + \frac{1}{2} \gamma^2 \frac{\partial^2 P}{\partial \xi_0^2} + \rho e^{\xi_0 + \bar{a}} \gamma \frac{\partial^2 P}{\partial x_0 \partial \xi_0} = 0, \quad (5.32)$$

we find that, for different $x_T$ values, the absolute values are all in the order of $10^{-7}$ or even smaller. In section 5.3.2 we come back to the discussion concerning the goodness of our approximation.

According to the discussion of Section 5.1, an extension of this model to the one with price jumps is straightforward: the new marginal probability distribution would be:

$$P_{EJ}(x_T|0, \zeta_0) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi} e^{ip(x_T-x_0-rT)+B(e^{\zeta_0-1})} \times e^{\frac{\beta(\zeta_0-\omega)}{2}\left[ \frac{\beta-\omega+1}{2} + \frac{\gamma^2}{2z^2} \right] T} \times \sqrt{1 + \frac{1-e^{-2\omega T}}{2\omega} \left[ \beta - \omega + B\gamma^2 \right]} \times e^{\lambda T \int_{-\infty}^{+\infty} [e^{-ipJ-1+ip(\epsilon^f-1)}] \varphi(J) dJ}, \quad (5.33)$$

where the same notations as in Eq.(5.28) are used.

### 5.3 European vanilla option pricing

#### 5.3.1 General pricing formulas

If we denote the general marginal propagator by

$$P(x_T|x_0, \sigma_0) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi} e^{ip(x_T-x_0-rT)} F(p, T) e^{U(p, T)}, \quad (5.34)$$
then the option pricing formula of a vanilla call option $C$ with expiration date $T$ and strike price $K$ is given by the discounted expectation value of the payoff:

$$C = e^{-rT} \int_{-\infty}^{+\infty} (e^{xT} - K)_+ \mathcal{P}(x_T|x_0, \sigma_0) dx_T$$

$$= \frac{G(0)}{2} + i \int_{-\infty}^{+\infty} \frac{dp \ e^{ip\ln\frac{K}{S_0} - rT} G(p)}{2\pi p}, \quad (5.35)$$

where

$$G(p) = S_0 F(p + i, T) e^{U(p+T)} - K e^{-rT} F(p, T) e^{U(p,T)}, \quad (5.36)$$

and $(x)_+ = \max(x, 0)$.

Here we have followed the derivation outlined in Ref. [214]. In particular for the EV model $F(p, T)$ equals:

$$F(p, T) = e^{\beta \omega_0 + \frac{J^2}{2\sigma_0^2} \left[ \frac{\beta - \omega + B\gamma^2}{\omega} \right] e^{\nu_0 T} - 1 + \frac{i\nu}{\sqrt{\omega} \left[ 1 - e^{-2\omega T} \right]}} \times e^{A(J) + B(J) \Theta(J)}, \quad (5.37)$$

At this stage one needs to specify the probability density function (PDF) for the jump sizes. Merton [12] and Kou [13] proposed a normal distributed jump size, denoted by $\varpi_M(J)$, and an asymmetric double exponential distributed one, denoted by $\varpi_K(J)$, respectively:

$$\varpi_M(J) = \frac{1}{\sqrt{2\pi}\delta} e^{-\frac{(J-\nu)^2}{2\delta^2}}, \quad (5.38)$$

$$\varpi_K(J) = \frac{p_+}{\eta_+} e^{\frac{\eta_+}{\eta_+} J} \Theta(J) + \frac{p_-}{\eta_-} e^{\frac{-\eta_+}{\eta_-} J} \Theta(-J). \quad (5.39)$$

For the Merton model $\nu$ is the mean jump size and $\delta$ is the standard deviation of the jump size. For Kou’s model $0 < \eta_+ < 1, \eta_- > 0$ are means of positive and negative jumps respectively, $p_+$ and $p_-$ represent the probabilities of positive and negative jumps, $p_+ > 0, p_- > 0, p_+ + p_- = 1$ and $\Theta$ is the Heaviside function.

According to expression (5.11), it is easy to derive their corresponding $U(p, T)$’s:

$$U_M(p, T) = \lambda T \left[ e^{-ip\omega - \frac{1}{2}\delta^2 p^2} - 1 + ip \left( e^{ip\omega + \frac{1}{2}\delta^2} - 1 \right) \right], \quad (5.40)$$

$$U_K(p, T) = \lambda T \left[ \frac{p_+}{1 + ip\eta_+} + \frac{p_-}{1 - ip\eta_-} - 1 + ip \left( \frac{p_+}{1 - \eta_+} + \frac{p_-}{1 + \eta_-} - 1 \right) \right]. \quad (5.41)$$

Using expression (5.37) and results (5.40), (5.41) in formulas (5.35), (5.36) allows to find the price of the vanilla call option for the exponential Vasiček stochastic volatility with price jumps model.
5.3.2 Monte Carlo simulations

To test our analytical pricing formula for the EV model, we focus on the parameters that most strongly influence the approximation. To satisfy the assumption that quadratic fluctuations around the mean reversion level \( \bar{a} \) capture the behavior of the volatility well, the mean reversion speed \( \beta \) and the volatility \( \gamma \) of asset volatility are crucial.

The substitution \( \tau = \gamma^2 t \) transforms expression (5.17) into

\[
dz(\tau) = \frac{\beta}{\gamma^2} [\bar{a} - z(\tau)] d\tau + dB_2(\tau),
\]

showing that it is actually the parameter \( c = \frac{\beta}{\gamma^2} \) which determines whether the approximation will be good. For bigger \( c \) values the approximation \( z(t) \approx \bar{a} \) will be better.

As the correlation parameter \( \rho \) controls the skewness of spot returns, we will also consider the typical negative and positive skewed cases by taking values \(-0.5, 0, 0.5\) for this parameter. On the other hand, the constant interest rate \( r \) and the mean reversion level \( \bar{a} \) do not influence the accuracy of the result a lot, and we just assume them to be constant values: \( r = 0.015 \) and \( \bar{a} = -1.6 \approx \ln 0.2 \). These two parameters seem to be quite reasonable for the present European options.

![Figure 5.2: Panel (1) shows the European call option price in the EV model. The red curve is our analytical result and the black crosses are Monte Carlo simulations. Panel two shows the relative deviation of the analytical result from the MC result for the EV model expressed in percentages. Panels (3) and (4) show the same thing for the EV model with respectively Merton’s jump and Kou’s jump. Parameter values \( S_0 = 100, r = 0.015, T = 1, \beta = 5, \bar{a} = -1.6, \gamma = 0.5, \rho = -0.5, \lambda = 10, \nu = -0.01, \delta = 0.03, p_+ = 0.3, p_- = 0.7, \eta_+ = 0.02, \eta_- = 0.04 \) are used here.](image)

To get an idea of what is a reasonable range for \( c \), and since calibration values for the EV model are not available, we took calibration values from the literature [10,218] for the Heston model and fitted our model to the volatility distribution of
the Heston model with those parameters. For [10] we obtained $c \approx 7$ and for [218] $c \approx 18$. Therefore in Table I we used values for $\beta$ and $\gamma$ such that $c$ ranges from 4.08 up to 25. We calculated prices for $S_0 = 100$ and $K = 90, 100$ and 110.

The comparison of our analytical solution with the MC solution for a European call option in the EV model as shown in Table 5.1 suggests that for the above mentioned parameter values the relative errors are less than 3% and most of the time even less than 1%, which is acceptable when we take the typical bid-ask spread for European options into account. Here each MC simulation runs 20,000,000 times.

For the basic EV model we can conclude that we found an approximation valid up to 3% for parameter values $c > 7$ (We only checked values of $c < 25$, but for bigger $c$ the approximation will only become better), $-0.5 < \rho < 0.5$, $T < 1$ and $0.9 < K/S_0 < 1.1$.

Finally we consider the vanilla call option pricing in EV model combined with Merton’s and Kou’s jumps, respectively. Since the jump process is independent from the approximation we made, we do not investigate the goodness of our approximation as thoroughly as in the basic EV model (assuming that if it is good there it will be good here). Figure 5.3.2 illustrates our analytical result for the EV model as well as the relative errors between the analytical result and the MC simulations for the EV model, the EV model with Merton’s jump and the EV model with Kou’s jump. Each MC simulation runs 300,000,000 times. These results suggest that the approximation error is typically less than 2%. And due to the fact that whenever the degree of moneyness (the ratio of the strike price $K$ to the initial asset price $S_0$) is relatively high, the average bid-ask spread tends to be relatively high for call options [239], our analytical results can serve as an easy way to get a quick estimate that is normally accurate enough for many practical applications.

\section*{5.A Derivation of equations (5.10), (5.11)}

The proof starts by assuming that a solution for the propagator $P_J(x_T, \sigma_T, T|x_0, \sigma_0, 0)$ of the form (5.10) exists. Below we show that this assumption indeed leads to a solution, which in turn justifies the assumption.

Since \( \int_{-\infty}^{+\infty} \frac{dp}{2\pi} e^{ip(x_T-x_0)} \frac{\partial F(\sigma_T, \sigma_0, r, p, T)}{\partial T} \) equals the right hand side of Eq.(5.4) and the derivative operators $\frac{\partial}{\partial x_T}$ and $\frac{\partial}{\partial \sigma_T}$ have no effect on $e^{U(p,T)}$, it follows that:

\[
\frac{\partial}{\partial x_T} \left[ - \left( r - \frac{1}{2} \sigma^2_T \right) P_J \right] + \frac{1}{2} \frac{\partial^2}{\partial x_T^2} \left[ \sigma^2_T P_J \right] + \frac{\partial}{\partial \sigma_T} \left[ -A(T, \sigma_T) P_J \right] \\
+ \frac{1}{2} \frac{\partial^2}{\partial \sigma_T^2} \left[ B^2(T, \sigma_T) P_J \right] + \rho \frac{\partial^2}{\partial x_T \partial \sigma_T} \left[ \sigma_T B(T, \sigma_T) P_J \right] \\
= \int_{-\infty}^{+\infty} \frac{dp}{2\pi} e^{ip(x_T-x_0)} \frac{\partial F(\sigma_T, \sigma_0, r, p, T)}{\partial T} e^{U(p,T)}.
\]

Adding the term $\lambda m^j \frac{\partial}{\partial x_T} P_J$, which is given by

\[
\lambda \int_{-\infty}^{+\infty} \frac{dp}{2\pi} e^{ip(x_T-x_0)} F(\sigma_T, \sigma_0, r, p, T) e^{U(p,T)} \int_{-\infty}^{+\infty} (e^J - 1) \varphi(J) dJ,
\]

62
as well as the term \( \lambda \int_{-\infty}^{+\infty} [P_J(x_T - J) - P_J(x_T)] \varpi(J) dJ \), which is given by

\[
\lambda \int_{-\infty}^{+\infty} \frac{dp}{2\pi} e^{ip(x_T - x_0)} F(\sigma_T, \sigma_0, r, p, T) e^{U(p,T)} \int_{-\infty}^{+\infty} (e^{-ipJ} - 1) \varpi(J) dJ, \tag{5.45}
\]

the right hand side of Eq.(5.8) is expressed as

\[
\int_{-\infty}^{+\infty} \frac{dp}{2\pi} e^{ip(x_T - x_0)} \frac{\partial F(\sigma_T, \sigma_0, r, p, T)}{\partial T} e^{U(p,T)} \\
+ \int_{-\infty}^{+\infty} \frac{dp}{2\pi} e^{ip(x_T - x_0)} F(\sigma_T, \sigma_0, r, p, T) e^{U(p,T)} \\
\times \lambda \int_{-\infty}^{+\infty} \left[ e^{-ipJ} - 1 + ip(e^J - 1) \right] \varpi(J) dJ. \tag{5.46}
\]

This, of course should equal the left hand side of Eq.(5.8), which is given by

\[
\int_{-\infty}^{+\infty} \frac{dp}{2\pi} e^{ip(x_T - x_0)} \frac{\partial F(\sigma_T, \sigma_0, r, p, T)}{\partial T} e^{U(p,T)} \\
+ \int_{-\infty}^{+\infty} \frac{dp}{2\pi} e^{ip(x_T - x_0)} F(\sigma_T, \sigma_0, r, p, T) \frac{\partial e^{U(p,T)}}{\partial T}. \tag{5.47}
\]

Expression (5.46) equals (5.47) when

\[
\frac{\partial U(p,T)}{\partial T} = \lambda \int_{-\infty}^{+\infty} \left[ e^{-ipJ} - 1 + ip(e^J - 1) \right] \varpi(J) dJ, \tag{5.48}
\]

from which the result (5.11) for \( U(p, T) \) follows.
Table 5.1: Comparison of our approximative analytic pricing result and the MC simulation value for the EV model.

<table>
<thead>
<tr>
<th>Parameter values</th>
<th>MC value(a)</th>
<th>Approx.(b)</th>
<th>Relative error&lt;br&gt;(b - a)/a (%)</th>
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For the remaining parameters the values $S_0 = 100$, $r = 0.015$, $\bar{a} = -1.6$ and $T = 1$ were used here.
Chapter 6

Pricing bounds for discrete arithmetic Asian options within Lévy models*

Up till now we focused on deriving analytical formulas for the option price. This is not always possible. Nevertheless, as in physics problems it is possible to derive bounds eg. through variational principles as the Jensen-Feynman inequality. In particular, in this chapter we present a method to derive lower and upper bounds for discretely monitored arithmetic Asian options within Lévy models. We then apply our method to the following Lévy models: Kou’s model [13], Merton’s model [12], the normal inverse Gaussian (NIG) model [118], the variance gamma (VG) model [113] and the CGMY model [109].

We contribute to the existing work by deriving bounds for discrete Asian options within a general Lévy model. The derivation of the lower bound is a generalization of the method used in Ref. [160] in the Black-Scholes framework to the setting of a general Lévy process. As was also the case in the Black-Scholes setting, the lower bound turns out to be very accurate. Furthermore this bound is very fast to evaluate, since for this purpose we only need to find the root of a monotonic function and evaluate a numerical integral of a smooth function composed out of elementary functions. For the upper bound we discuss two approaches. The derivation of the first upper bound (UB) is based on the upper bound presented in Ref. [180] which is an improvement of the upper bound presented in Ref. [160], both papers are restricted to the Black-Scholes framework. This upper bound is associated to the lower bound as will be explained later. For at- and in-the-money options† this upper bound, together with the lower bound, is accurate enough for pricing the Asian option. Moreover, when the strike converges to zero both the lower and this upper bound converge to the exact price. For out of the money options this upper bound becomes worse. The second upper bound (UBC) is based on the theory of comonotonicity explained in detail in Refs. [178,179]. In these papers this bound is

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*This chapter is based on the article [240], which is joint work with Ling Zhi Liang, Jacques Tempère and Ann De Schepper.

†At-the-money options are options for which the strike price equals the spot price. A call/putt option is in-the-money when the strike price is below/above the spot price. For an out-of-the-money option the opposite is true.
stated for a rather general model, nevertheless this bound has only been explicitly calculated for the Black-Scholes model. In Ref. [180] these bounds are among others compared within the Black-Scholes setting, with the conclusion that UB is better than UBC except for far out of the money options. For the Lévy models that we consider in this chapter we also conclude that UB is more accurate than UBC, we do not inquire into the far out of the money strikes for which the UBC result becomes more accurate than the UB result in the Black-Scholes framework. For Kou’s model, Merton’s model, the NIG model and the CGMY model we compare our bounds with the numerical results of Ref. [184]. For the VG model we compare with the lower bound of Ref. [186] and with a Monte Carlo simulation.

The chapter is organized as follows. In section 6.1 we discuss the model for the underlying process. In section 6.2 our lower bound for the arithmetic Asian option for a general Lévy model is derived. In section 6.3 we derive the corresponding upper bound and discuss the comonotonic upper bound. In section 6.4 we apply these results to Kou’s model, Merton’s model, the NIG model, the VG model and the CGMY model. Some of the derivations are added in the appendix.

6.1 Setting the stage

The asset price at time \( t \) will be denoted by \( S_t \). The arithmetic average \( A \) is defined by

\[
A = \frac{1}{N + 1} \sum_{k=0}^{N} S_{t_k}, \tag{6.1}
\]

with \( t_k = k\Delta \). It is straightforward to adapt the results to other definitions of the average such as \( A = \frac{1}{N} \sum_{k=1}^{N} S_{t_k} \). The demeaned logreturn \( X_t \) is related to \( S_t \) by the following formula

\[
S_t = S_0 \exp \left[ (r - m) t + X_t \right], \tag{6.2}
\]

where \( r \) is the risk free interest rate and the drift parameter \( m \) will be specified later. The average of \( X \) will be denoted by:

\[
\bar{X}_T = \frac{1}{N + 1} \sum_{k=0}^{N} X_{t_k} = \frac{1}{N + 1} \sum_{k=1}^{N} X_{t_k}. \tag{6.3}
\]

This last equality follows because \( X_0 = 0 \). Denote

\[
X_{t_k}^\Delta = X_{t_k} - X_{t_{k-1}}. \tag{6.4}
\]

For simplicity of notation \( S_{t_k} \) will sometimes be written as \( S_k \) and \( X_{t_k} \) as \( X_k \). The basic assumption for our method is that the logarithm of the asset price follows a Lévy process. This implies that non-overlapping increments are independent and increments over equal time spans are identically distributed. The characteristic function \( \phi_{X_{t_k}^\Delta} \) associated with the distribution of the increments \( X_{t_k}^\Delta \) is defined by:

\[
\phi_{X_{t_k}^\Delta} (w) = \int_{-\infty}^{+\infty} \exp (iwx) P(x) \, dx, \tag{6.5}
\]
with $P(x)$ the density function of $X^\Delta_{tk}$. For later use in the calculations, it turns out to be convenient to introduce the function $f(w)$ defined by the following relation:

$$\phi_{X^\Delta_{tk}}(w) = \exp \left[ f(w) \Delta \right]. \quad (6.6)$$

Note that the characteristic function $\phi_{X^\Delta_{tk}}$ does not depend on $k$ since all the increments are identically distributed. We didn’t determine the drift parameter $m$ yet. One way to determine $m$ is to require that the asset price discounted by $B_t = e^{rt}$ is a martingale:\footnote{More information on equivalent martingale measures for Lévy processes can be found in Refs. [105, 106, 241, 242].}

$$E \left[ \frac{S_T}{B_T} \right] = \frac{S_0}{B_0}, \quad \forall T > 0.$$ The previous expectation value is given by:

$$E \left[ \frac{S_T}{B_T} \right] = E \left[ S_0 e^{-mT} e^{X_T} \right] = S_0 e^{-mT} \int e^x P(x) dx = e^{-mT} \int e^x \left( \int e^{-i\omega x + f(\omega)T} \frac{d\omega}{2\pi} \right) dx,$$

(6.7)

where the probability density of $x$ is written as a Fourier transform. If we change the order of the integrals:

$$E \left[ \frac{S_T}{B_T} \right] = S_0 \exp (-mT) \int \exp (f(\omega)T) \int \exp (-i(\omega+i)x) dx \frac{d\omega}{2\pi}, \quad (6.8)$$

then we recognize the Fourier transform of the delta function:

$$E \left[ \frac{S_T}{B_T} \right] = S_0 \exp (-mT) \int \exp (f(\omega)T) \delta(\omega+i) d\omega. \quad (6.9)$$

This integral gives as result:

$$E \left[ \frac{S_T}{B_T} \right] = S_0 \exp (-mT) \exp (f(-i)T). \quad (6.10)$$

To satisfy the condition that $E \left[ \frac{S_T}{B_T} \right] = S_0$, it follows that:

$$m = f(-i). \quad (6.11)$$

### 6.2 The lower bound

In this section we will derive a lower bound for the call price $C$ of a discretely monitored arithmetic Asian option. Recall that $C$ is given by:

$$C = \exp (-rT) E \left[ (A - K)_+ \right] = \exp (-rT) E \left[ E \left[ (A - K)_+ | X_T \right] \right], \quad (6.12)$$
with \((x)_+ = \max(x, 0)\), \(E[X]\) is the expectation value of \(X\) and \(E[X|Z]\) is the conditional expectation value of \(X\) with respect to \(Z\). Since \((.)_+\) is a convex function the inequality of Jensen \[243\] learns that \(C\) satisfies:

\[
C \geq \exp \left( -rT \right) E \left[ (E[A|\bar{X}_T] - K)_+ \right].
\]

If in the expression above \(A\) is written in full with the aid of equation (6.1) and then \(S_k\) is substituted by expression (6.2) we obtain

\[
C \geq e^{-rT} E \left[ \left( \frac{S_0}{N+1} \sum_{k=0}^{N} e^{(r-m)k\Delta} e^{X_k} \mid \bar{X}_T = \bar{x} \right) \right]_+.
\]

If the conditional expectation changes places with the sum and if the expectation over \(\bar{X}_T\) is written out explicitly we get:

\[
C \geq e^{-rT} \int_{-\infty}^{+\infty} \left[ \frac{S_0}{N+1} \left( 1 + \sum_{k=1}^{N} e^{(r-m)k\Delta} E \left[ e^{X_k} \mid \bar{X}_T = \bar{x} \right] \right) - K \right]_+ P_{\bar{X}_T}(\bar{x}) d\bar{x}.
\]

\(P_{\bar{X}_T}(\bar{x})\) is calculated in Ref. \[184\] and is given by:

\[
P_{\bar{X}_T}(\bar{x}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp \left( -i\omega_1 \bar{x} \right) \phi_{X,T}(\omega_1) d\omega_1,
\]

with

\[
\phi_{X,T}(\omega_1) = \exp \left( \sum_{k=1}^{N} \left( \omega_1 \left( \frac{N-k+1}{N+1} \right) \right) \right),
\]

where \(f\) was introduced in expression (6.6). The conditional expectation \(E[e^{X_k}|\bar{X}_T]\) present in equation (6.15) is equal to:

\[
E \left[ \exp (X_k) \mid \bar{X}_T = \bar{x} \right] = \int_{-\infty}^{+\infty} \exp (x_k) P(x_k, \bar{x}) dx_k.
\]

The joint probability distribution function \(P(x_k, \bar{x})\) can be obtained in a similar way as the \(P(\bar{x})\), the derivation is given in appendix 6.A. The result is given by:

\[
P(\bar{x}, x_k) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp \left( -i\omega_1 \bar{x} - i\omega_2 x_k \right) \phi_{X_k,X}(\omega_1, \omega_2) d\omega_1 d\omega_2,
\]

with

\[
\phi_{X_k,X}(\omega_1, \omega_2) = \exp \left[ \left( \sum_{l=1}^{k} f \left( \omega_1 \left( \frac{N-l+1}{N+1} \right) + \omega_2 \right) + \sum_{l=k+1}^{N} f \left( \omega_1 \left( \frac{N-l+1}{N+1} \right) \right) \Delta \right) \right].
\]
Substituting (6.19) into (6.18), \( E \left[ \exp \left( X_k \right) | \bar{X}_T = \bar{x} \right] \) becomes:

\[
E \left[ \exp \left( X_k \right) | \bar{X}_T = \bar{x} \right] = \frac{1}{4\pi^2 P_{\bar{X}_T}(\bar{x})} \int_{-\infty}^{+\infty} e^{x_k} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-i\omega_1 \bar{x} - i\omega_2 x_k} \phi_{X_k, \bar{X}_T}(\omega_1, \omega_2) \, d\omega_1 \, d\omega_2 \, dx_k
\]

\[
= \frac{1}{4\pi^2 P_{\bar{X}_T}(\bar{x})} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-i\omega_1 \bar{x} - i(\omega_2 + i)x_k} \, dx_k \phi_{X_k, \bar{X}_T}(\omega_1, \omega_2) \, d\omega_1 \, d\omega_2
\]

\[
= \frac{1}{2\pi P_{\bar{X}_T}(\bar{x})} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \delta(\omega_2 + i) e^{-i\omega_1 \bar{x}} \phi_{X_k, \bar{X}_T}(\omega_1, \omega_2) \, d\omega_1 \, d\omega_2
\]

\[
= \frac{1}{2\pi P_{\bar{X}_T}(\bar{x})} \int_{-\infty}^{+\infty} e^{-i\omega_1 \bar{x}} \phi_{X_k, \bar{X}_T}(\omega_1, -i) \, d\omega_1.
\]

(6.21)

An efficient way to evaluate expression (6.15) is to first determine the value for \( \bar{x} \) for which the monotonically increasing function

\[
\frac{S_0}{(N + 1)} \left( 1 + \frac{1}{2\pi P_{\bar{X}_T}(\bar{x})} \sum_{k=1}^{N} e^{(r-m)k\Delta} \int_{-\infty}^{+\infty} e^{-i\omega_1 \bar{x}} \phi_{X_k, \bar{X}_T}(\omega_1, -i) \, d\omega_1 \right) - K
\]

becomes zero, denote this value \( a \). Doing so inequality (6.15) can be rewritten as:

\[
C \geq e^{-rT} \int_{a}^{\infty} \left[ \frac{S_0}{N + 1} \left( 1 + \sum_{k=1}^{N} e^{(r-m)k\Delta} E \left[ \exp \left( X_k \right) | \bar{X}_T = \bar{x} \right] \right) dt \right] \, d\bar{x}.
\]

(6.23)

This can be simplified to (see appendix 6.B)

\[
C \geq \exp \left( -rT \right) \left[ \frac{g(0)}{2} - \frac{i}{2\pi} \int_{-\infty}^{+\infty} \exp \left( -i\omega_1 a \right) g(\omega_1) - g(0) \, d\omega_1 \right],
\]

(6.24)

with

\[
g(\omega_1) = \frac{S_0}{(N + 1)} \left( \phi_{X_1}(\omega_1) + \sum_{k=1}^{N} e^{(r-m)k\Delta} \phi_{X_k, \bar{X}_T}(\omega_1, -i) \right) - K\phi_{\bar{X}_T}(\omega_1).
\]

(6.25)

Formula (6.24) is our central result concerning the lower bound, as already mentioned this formula is fast to evaluate and is very accurate, which will become more clear in section 6.4.

### 6.3 Upper bound

In this section we will first derive an upper bound which supplements the lower bound in the sense that this upper bound arises by calculating an upper bound for the difference between the real price and the lower bound. Next we will briefly discuss how to calculate the comonotonic upper bound in the setting of Lévy models.
6.3.1 Upper bound corresponding to the lower bound.

In this section the lower bound of the previous section will be supplemented by an upper bound. This makes it possible to reliably qualify the call price \( C \). The upper bound is found by determining the error \( \varepsilon \) made when calculating the lower bound, i. e.:

\[
\varepsilon = \exp(-rT) \left\{ E \left( (A - K)_+ \right) - E \left( E[A|\bar{X}_T] - K \right)_+ \right\}. \tag{6.26}
\]

In Refs. [160, 180] it is shown that this is dominated by:

\[
\varepsilon \leq \frac{\exp(-rT)}{2} E \left[ \sqrt{\text{Var} \left( \frac{1}{N+1} \sum_{k=0}^{N} S_k - K \right| \bar{X}_T } \right]. \tag{6.27}
\]

with

\[
b = \ln \left( \frac{K}{S_0} \right) + \frac{m - r}{2} T \tag{6.28}
\]

and \( \theta(\cdot) \) the Heaviside step function, for completeness the derivation of expression (6.27) is included in appendix 6.C. Denote the right hand side of inequality (6.27) by \( \varepsilon' \). It is easily seen that the derivation given in Refs. [160, 180] to go from formula (6.26) to formula (6.27) is model independent. In other words their derivation applies to our situation without any essential changes. The upper bound is then given by:

upper bound = lower bound + \( \varepsilon' \). A straightforward calculation shows that the variance in formula (6.27) is equal to:

\[
\text{Var} \left( \frac{1}{N+1} \sum_{k=0}^{N} S_k - K \right| \bar{X}_T \right) = \frac{S_0^2}{(N+1)^2} \times
\]

\[
\left( 2 \sum_{k=2}^{N} e^{(r-m)k\Delta} \sum_{l=1}^{k-1} e^{(r-m)l\Delta} \left( E \left[ e^{X_k + X_l} | \bar{X} \right] - E \left[ e^{X_k} | \bar{X} \right] E \left[ e^{X_l} | \bar{X} \right] \right) \right). \tag{6.29}
\]

In formula (6.21) an expression for \( E \left[ \exp(X_k) \right| \bar{X}_T = \bar{x} \) is given. In a similar way \( E \left[ \exp(2X_k) \right| \bar{X}_T = \bar{x} \) equals:

\[
E \left[ \exp(2X_k) \right| \bar{X}_T = \bar{x} \right] = \frac{1}{2\pi P_{\bar{X}_T} (\bar{x})} \int_{-\infty}^{+\infty} \exp(-i\omega_1 \bar{x}) \phi_{X_k, \bar{X}_T}(\omega_1, -2i) d\omega_1. \tag{6.30}
\]

Also \( E \left[ \exp(X_k + X_l) \right| \bar{X}_T \) can be calculated in a similar way, though first \( P(x_k, x_l, \bar{x}) \) needs to be derived. This can be done in the same way as \( P(x_k, \bar{x}) \) was derived (see appendix 6.A), the result for \( P(x_k, x_l, \bar{x}) \) with \( l < k \) is given by:

\[
P(x_k, x_l, \bar{x}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-i\omega_1 \bar{x} - i\omega_2 x_k - i\omega_3 x_l} \phi_{X_k, X_l, \bar{X}_T}(\omega_1, \omega_2, \omega_3) d\omega_1 d\omega_2 d\omega_3. \tag{6.32}
\]
with
\[ \phi_{X_k, X_l, \bar{x}_T} (\omega_1, \omega_2, \omega_3) = \exp [\Omega \Delta]. \] (6.33)

and
\[
\Omega = \sum_{k_2=1}^{l} f \left( \frac{1}{N+1} (N - k_2 + 1) + \omega_2 + \omega_3 \right) + \sum_{k_2=l+1}^{k} f \left( \frac{1}{N+1} (N - k_2 + 1) + \omega_2 \right) + \sum_{k_2=k+1}^{N} f \left( \frac{1}{N+1} (N - k_2 + 1) \right) \] (6.34)

This allows the calculation of \( E \left[ \exp (X_k + X_l) | \bar{x}_T = \bar{x} \right], \) which results in:
\[
E \left[ e^{X_k + X_l} | \bar{x}_T = \bar{x} \right] = \frac{1}{2\pi P_{X_T}(\bar{x})} \int_{-\infty}^{+\infty} e^{-i\omega_1 \bar{x}} \phi_{X_k, X_l, \bar{x}_T} (\omega_1, -i, -i) d\omega_1. \] (6.35)

Combining formulas (6.21), (6.29), (6.30) and (6.35) expression (6.27) becomes:
\[
\varepsilon' = \frac{S_0 e^{-rT}}{2(N+1)} \int_{-\infty}^{b} \sqrt{h(\bar{x})} d\bar{x}. \] (6.36)

with
\[
h(\bar{x}) = 2 \sum_{k=2}^{N} e^{(r-m)k\Delta} \sum_{l=1}^{k-1} e^{(r-m)l\Delta} \chi \left( \begin{array}{c}
\frac{P_{X_T}(\bar{x})}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega_1 \bar{x}} \phi_{X_k, X_l, \bar{x}_T} (\omega_1, -i, -i) d\omega_1 - \\
\frac{1}{4\pi^2} \int_{-\infty}^{+\infty} e^{-i\omega_1 \bar{x}} \phi_{X_k, X_l, \bar{x}_T} (\omega_1, -i) d\omega_1 + \int_{-\infty}^{+\infty} e^{-i\omega_1 \bar{x}} \phi_{X_k, X_l, \bar{x}_T} (\omega_2, -i) d\omega_2 \\
+ \sum_{k=1}^{N} e^{2(r-m)k\Delta} \left( \frac{P_{X_T}(\bar{x})}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega_1 \bar{x}} \phi_{X_k, X_l, \bar{x}_T} (\omega_1, -2i) d\omega_1 \right) \right)
\] (6.37)

Substituting expression (6.16) and changing places of the summations and the integrations the above expression becomes:
\[
\varepsilon' = \frac{S_0}{2\pi(N+1)} \int_{-\infty}^{b} d\bar{x} \sqrt{\int_{-\infty}^{+\infty} e^{-i(\omega_1 + \omega_2) \bar{x}} \tilde{g}(\omega_1, \omega_2) d\omega_1 d\omega_2} \] (6.38)

with
\[
\tilde{g}(\omega_1, \omega_2) = \sum_{k=2}^{N} e^{(r-m)k\Delta} \sum_{l=1}^{k-1} e^{(r-m)l\Delta} \left( \phi_{X_T}(\omega_2) \phi_{X_k, X_l, \bar{x}_T} (\omega_1, -i, -i) \right) \left( -\phi_{X_k, X_l, \bar{x}_T} (\omega_1, -i) \phi_{X_1, X_T} (\omega_2, -i) \right) + \frac{1}{2} \sum_{k=1}^{N} e^{2(r-m)k\Delta} \left[ \phi_{X_T}(\omega_2) \phi_{X_k, X_T} (\omega_1, -2i) - \phi_{X_k, X_T} (\omega_2, -i) \phi_{X_k, X_T} (\omega_1, -i) \right]. \] (6.39)
Formula (6.38) is our central result for the upper bound. The accuracy and the evaluation time depends from model to model. Although the result for the lower bound concerning both the accuracy and the evaluation time is much better, for in and at the money options the accuracy is sufficient to calculate the price of Asian options in an acceptable time for most of the investigated models.

### 6.3.2 Comonotonic upper bound

This bound is derived in Refs. [178,179] for a rather general model. Nevertheless this bound has not yet been calculated for the Lévy models considered in this chapter. Since the method explained in detail in Refs. [178,179] applies to the situation of Lévy models directly, we only give the formulas necessary to calculate the bound within Lévy models. The recipe to calculate this upper bound goes as follows. To start with, one has to be able to evaluate the the inverse cumulative distribution function of the random variable $S_k$, denoted here by $F_{S_k}^{-1}(u)$. As a first step this function can be expressed as a function of the inverse cumulative distribution function of the random variable $X_k$:

$$F_{S_k}^{-1}(u) = F_{S_0 \exp((r-m)k\Delta + X_k)}^{-1}(u),$$

$$= S_0 \exp[(r-m)k\Delta + F_{X_k}^{-1}(u)].$$

(6.40)

Analogously to the derivation of formula (6.24), presented in appendix 6.B the cumulative distribution function of $X_k$ can be shown to equal:

$$F_{X_k}(x) = \frac{e^{f(0)k\Delta}}{2} + \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-i\omega x + f(x)k\Delta} - e^{f(0)k\Delta}}{\omega} dw.$$  

(6.41)

The inversion of this formula needs to be done numerically. Then it needs to be numerically determined for which value $u$ the following equality holds:

$$\sum_{k=1}^{N} F_{S_k}^{-1}(u) = (N + 1) \left( K - \frac{S_0}{N + 1} \right).$$

(6.42)

With this value for $u$ the upper bound for the Asian call price $C$ is given by:

$$C \leq \frac{e^{-rT}}{N + 1} \sum_{k=1}^{N} E \left( [S_k - F_{S_k}^{-1}(u)]_+ \right).$$

(6.43)

Although equations (6.42) and (6.43) form an elegant presentation of the comonotonic upper bound, it is hard to implement this for a general model. For the Black-Scholes model it is quite straightforward to determine which integration grid should be used in formula (6.41) for every $k$. For Lévy models on the contrary this can be a tedious task. For some models it might be preferable to calculate the cumulative distribution function of $X_k$ by numerically integrating its probability distribution function.
6.4 Results for several Lévy models

In this section the bounds derived in the previous sections will be applied to several Lévy models. The parameter values we use in this section are chosen in order to be able to compare with existing work.

For Kou’s model, Merton’s model, the normal inverse Gaussian model and the CGMY model no analytical bounds have been calculated as far as we know and therefore we compare with the numerical results of Fusai and Meucci [184] and use the same parameter values as are used there, we will refer to these results as the FM results. The parameter values of Ref. [184] are taken from Ref. [105] where these values are obtained by calibrating on a set of European call options on the S&P500 index, which guarantees the relevance of these parameter values.

For the variance gamma model a lower bound has been calculated in Ref. [186], accordingly we use the parameter values of Table 8 of this reference. For these parameter values our lower bound is considerably more accurate than the lower bound presented in Ref. [186]. Regarding the method of Ref. [186] it is important to mention that it has a wider scope than our method, for it can be used model independent. With this method a lower bound for the option price can be found based on prices of European call options assumed to be observable in the market.

Having compared our results with the numerical results of Ref. [184] for the first four Lévy models and with the lower bound of Ref. [186] for the variance gamma model, we also compare for all the models our bounds with Monte Carlo simulations and with the comonotonic upper bound. For the first four models the Monte Carlo results were taken from Ref. [184]. For the variance gamma model we carried out a Monte Carlo simulation of $4 \times 10^8$ iterations. For all the models and strikes used here our upper bound seems to be significantly more accurate than the comonotonic upper bound.

For the first four models the lower bound turns out to be very accurate and much better than the upper bound. This is in agreement with the known results for the Black-Scholes case. For the variance gamma model the difference in accuracy between the lower and upper bound is less pronounced, although for most strikes the lower bound is still more accurate. This difference is most likely not due to the model. It is probably the consequence of a longer time to maturity, and the fact that the parameter values used here do not match the parameter values used for the other models.

Finally, note that based on the lower and upper bound two natural choices for an approximation arise: if no other information than the bounds is available the average of the bounds is the most obvious choice since it minimizes the maximal possible error. However, since it is known from the Black-Scholes setting that the lower bound is often better than the upper bound, the lower bound is a natural second choice for an approximation. Therefore for every model we also compute the relative errors made with these two approximations.
6.4.1 Kou’s model

For this model the function \( f(w) \) that specifies the model, can be derived from results in Ref. [13]:

\[
f(w) = -\frac{\sigma^2}{2}w^2 + \lambda \left( \frac{(1-p)\eta_2}{\eta_2 + i\omega} + \frac{p\eta_1}{\eta_1 - i\omega} - 1 \right).
\] (6.44)

Here \( \sigma \) is the diffusion volatility, \( \lambda \) is the jump intensity, \( p, 1-p \) is the probability of an up, down jump respectively and \( \eta_1, \eta_2 \) control the exponentially distributed up and down jump sizes (cf. section 5.3.1). The function \( g(\omega_1) \) necessary for the lower bound and given by (6.25) becomes:

\[
g(\omega_1) = \exp \left( -\frac{\sigma^2T}{2} \omega_1^2 \frac{2N + 1}{6(N + 1)} - \lambda T + \lambda \Delta \sum_{l=1}^{N} \Omega (\omega_1, l, 0) \right) \left( \frac{S_0}{N + 1} \right) \times \left\{ 1 + \sum_{k=1}^{N} \exp \left[ \left( r\lambda k + i\sigma^2 \omega_1 \frac{(2N-k+1)k}{2(N+1)} - \lambda k [\Omega (0, 0, -i) - \Omega (\omega_1, l, -i) - \Omega (\omega_1, l, 0)] \right) \right] \right\} - K),
\] (6.45)

with

\[
\Omega (\omega, l, a) = \frac{(1-p)\eta_2}{\eta_2 + i\left( \omega \frac{1}{N+1} \right) + a} + \frac{p\eta_1}{\eta_1 - i\left( \omega \frac{1}{N+1} + a \right)}.
\] (6.46)

After substituting the characteristic functions in formula (6.39) with their explicit expressions (i.e. formulas (6.17), (6.20) and (6.33)) and rearranging terms, \( \tilde{g}(\omega_1, \omega_2) \) becomes:

\[
\tilde{g}(\omega_1, \omega_2) = \exp \left( -\frac{\sigma^2T}{2} \omega_1^2 \frac{2N + 1}{6(N + 1)} - 2\lambda T + \tilde{f}(\omega_1, \omega_2, 0, 0, 1, N) - i\sigma^2 \omega_1 \frac{T}{2} \right) \times \sum_{k=1}^{N} \exp \left( h(k) + \tilde{f}(\omega_1, \omega_1, 1, 0, -1, k) \right) \left( \sum_{l=1}^{k-1} q(l) + \frac{1}{2} q(k) \right),
\] (6.47)

with

\[
q(k) = e^{h(k)} \left( \begin{array}{c}
\exp \left( \sigma^2 \Delta k + \tilde{f}(\omega_1, \omega_1, 2, 1, -1, k) \right) \\
- \exp \left( \tilde{f}(\omega_2, \omega_2, 1, 0, -1, k) \right)
\end{array} \right),
\] (6.48)

\[
h(k) = \left[ r - \lambda \left( \frac{(1-p)\eta_2}{\eta_2 + 1} + \frac{p\eta_1}{\eta_1 - 1} - 1 \right) \right] \Delta k,
\] (6.49)

and

\[
\tilde{f}(\omega_1, \omega_2, a, b, c, k) = i\sigma^2 \omega_1 \frac{(2N-k+1)k}{2(N+1)} \Delta
\]

\[
+ \lambda \Delta \sum_{k_2=1}^{k} \left( \frac{(1-p)\eta_2}{\eta_2 + a + i\omega_2} \frac{2N - k + 1}{N+1} + \frac{c}{\eta_1 - b - i\omega_2} \frac{2N - k + 1}{N+1} \right) \nonumber.
\] (6.50)

The results for this model are given in Table 6.1.
Table 6.1: In this table our bounds for prices of arithmetic Asian call options with strike $K$ are shown for Kou’s model. Our lower bound (LB) and upper bound (UB) are compared with the FM and Monte Carlo results (MC) of Ref. [184] and with the comonotonic upper bound (UBC). The parameter values used are: $S_0 = 100$, $T = 1$, $r = 0.0367$, $\sigma = 0.120381$, $\lambda = 0.330966$, $p = 0.2071$, $\eta_1 = 9.65997$, $\eta_2 = 3.13868$. MR stands for the maximal relative error which is made when the average of the lower and the upper bound is used as an approximation. RLB stands for the relative error compared with the Monte Carlo result which is made when the lower bound is used as an approximation.

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6.4.2 Merton’s model

For this model $f(w)$ can be derived from results in Ref. [12]:

$$f(w) = -\frac{\sigma^2}{2}w^2 + \lambda \left[ \exp \left( i\omega \alpha - \frac{\omega^2 \delta^2}{2} \right) - 1 \right]. \quad (6.51)$$

The parameters $\sigma$ and $\lambda$ are again the diffusion volatility and the jump intensity, $\alpha$ and $\delta$ are the mean and the standard deviation of the jump size. The results for Kou’s model are easily adapted to Merton’s model. The function $g(\omega_1)$ becomes:

$$g(\omega_1) = \exp \left\{ -\frac{\sigma^2 T}{2} \omega_1^2 \frac{(2N + 1)}{6(N + 1)} - \lambda T + \lambda \Delta \sum_{l=1}^{N} \Omega(\omega_1, l, 0) \right\} \left( \frac{S_0}{(N + 1) \times} \right)$$

$$\left[ 1 + \sum_{k=1}^{N} \exp \left\{ i r k - \lambda \Delta \left[ \Omega(0, 0, -i) - 1 \right] + i \sigma^2 \omega_1 \left( \frac{2(N + k + 1)}{2(N + 1)} \right)^k \right] \right] - K \right) \right\}, \quad (6.52)$$

with this time

$$\Omega(\omega, l, b) = \exp \left( i \left( \frac{\omega (N - l + 1)}{N + 1} + b \right) \alpha - \frac{(\omega (N - l + 1) + b)^2}{2} \delta^2 \right). \quad (6.53)$$
The function \( \tilde{g} (\omega_1, \omega_2) \), Eq. (6.39) becomes for Merton’s model:
\[
\tilde{g} (\omega_1, \omega_2) = 
\exp \left[ -\frac{\sigma^2 T}{2} \left( w_2^2 + \omega_1^2 \right) \frac{2N + 1}{6(N + 1)} - 2\lambda T + f (\omega_1, \omega_2, 0, 0, 1, N) - i\sigma^2 \omega_1 \frac{T}{2} \right] 
\times \sum_{k=1}^{N} \exp \left( h (k) + f (\omega_1, \omega_1, -i, 0, -1, k) \right) \left( \sum_{l=1}^{k-1} q (l) + \frac{1}{2} q (k) \right),
\]
where
\[
h (k) = \left[ r - \lambda \left( \exp \left( \alpha + \frac{\delta^2}{2} \right) - 1 \right) \right] \Delta k,
\]
\[
q (k) = e^{h(k)} \left( \exp [\sigma^2 \Delta k + f (\omega_1, \omega_1, -2i, -i, -1, k)] - \exp [f (\omega_2, \omega_2, -i, 0, -1, k)] \right),
\]
and
\[
f (\omega_1, \omega_2, a, b, c, k) = i\sigma^2 \omega_1 \frac{(2N - k + 1) k \Delta}{2(N + 1)}
+ \lambda \Delta \sum_{k_2=1}^{k} \exp \left( i \left( \omega_1 \frac{N-k_2+1}{N+1} + a \right) \alpha - \left( \omega_1 \frac{N-k_2+1}{N+1} + a \right) \frac{\delta^2}{2} \right)
+ c \exp \left( i \left( \omega_2 \frac{N-k_2+1}{N+1} + b \right) \alpha - \left( \omega_2 \frac{N-k_2+1}{N+1} + b \right) \frac{\delta^2}{2} \right).
\]

The numerical performance of our bounds for the Merton model are illustrated in table 6.2.

6.4.3 The NIG model

For this model \( f (w) \) can be derived from results in Ref. [118]:
\[
f (w) = -\delta \left( \sqrt{\alpha^2 - (\beta + i\omega)^2} - \sqrt{\alpha^2 - \beta^2} \right).
\]

Here \( \delta \) is the scale parameter, \( \alpha \) governs the tail heaviness and \( \beta \) determines the asymmetry of the distribution. The function \( g (\omega_1) \) becomes:
\[
g (\omega_1) = \exp \left\{ \delta T \sqrt{\alpha^2 - \beta^2} - \delta \Delta \sum_{l=1}^{N} \Omega (\omega_1, l, 0) \right\} \left( \frac{S_0}{(N + 1)} \right) \times \left\{ 1 + \sum_{k=1}^{N} \exp \left[ \left( r k + \delta k \left[ \Omega (0, 0, -i) - \sqrt{\alpha^2 - \beta^2} \right] \Delta \right) \right] \right\} - K,
\]
with now
\[
\Omega (\omega, l, b) = \sqrt{\alpha^2 - \left[ \beta + i \left( \omega \frac{(N - l + 1)}{N + 1} + b \right) \right]^2}.
\]
Table 6.2: In this table our bounds for prices of arithmetic Asian call options with
strike \( K \) for Merton’s model are shown. Our lower bound (LB) and upper bound
(UB) are compared with the FM and Monte Carlo results (MC) of Ref. [184] and with
the comonotonic upper bound (UBC). The parameter values used are:

\[ S_0 = 100, \quad T = 1, \quad r = 0.0367, \quad \sigma = 0.126349, \quad \alpha = -0.390078, \quad \lambda = 0.174814, \quad \delta = 0.338796. \]

MR stands for the maximal relative error which is made when the average of the lower
and the upper bound is used as an approximation. RLB stands for the relative error
compared with the Monte Carlo result which is made when the lower bound is used
as an approximation.

Furthermore, we get:

\[
\tilde{g}(\omega_1, \omega_2) = \exp \left[ 2\delta T \sqrt{\alpha^2 - \beta^2} + f(\omega_1, \omega_2, 0, 0, 1, N) \right] \times \\
\sum_{k=1}^{N} \exp \left( h(k) + f(\omega_1, \omega_1, -i, 0, -1, k) \right) \left( \sum_{l=1}^{k-1} q(l) + \frac{1}{2} q(k) \right),
\]  

(6.61)

with

\[
h(k) = \left[ r + \delta \left( \sqrt{\alpha^2 - (\beta + 1)^2} - \sqrt{\alpha^2 - \beta^2} \right) \right] \Delta k,
\]

(6.62)

\[
q(k) = \exp[h(k)] \left( \exp[f(\omega_1, \omega_1, -2i, -i, -1, k)] - \exp[f(\omega_2, \omega_2, -i, 0, -1, k)] \right),
\]

(6.63)

and

\[
f(\omega_1, \omega_2, a, b, c, k) = -\delta \Delta \sum_{k_1=1}^{k} \left( \frac{\sqrt{\alpha^2 - \left[ \beta + i \omega_1 \left( \frac{N-k_1+1}{N+1} + a \right) \right]^2}}{c} \frac{\sqrt{\alpha^2 - \left[ \beta + i \omega_2 \left( \frac{N-k_1+1}{N+1} + b \right) \right]^2}}{c} \right).
\]

(6.64)

The accuracy of our bounds for the NIG model and the parameter values discussed
in the beginning of this section is revealed in Table 6.3.

### 6.4.4 The CGMY model

For this model \( f(w) \) can be derived from results in Ref. [109]:

\[
f(w) = C \Gamma(-Y) \left( (M-i\omega)^Y + (G+i\omega)^Y - (G^Y + M^Y) \right).
\]

(6.65)
the fine structure of the stochastic process. The function $g$ determines the exponential decay on the right and the left of the Lévy density and $Y$ determines the fine structure of the stochastic process. The function $g(\omega_1)$ becomes:

$$g(\omega_1) = \exp \left( -CT(-Y)(G^Y + M^Y)T + CT(-Y)\Delta \sum_{l=1}^{N} \Omega(\omega_1, l, 0) \right) \left( \frac{S_0}{(N+1)} \times \left\{ 1 + \sum_{k=1}^{N} \exp \left( \left( \frac{rk - kCT(-Y)[\Omega(0,0,-i) - (G^Y + M^Y)] + CT(-Y)\sum_{l=1}^{k} [\Omega(\omega_1, l, -i) - \Omega(\omega_1, 0, l)] \Delta) \right) - K \right) \right\} \right),$$

with this time

$$\Omega(\omega, l, b) = \left( M - i \left( \omega \frac{(N - l + 1)}{N + 1} + b \right) \right)^Y + \left( G + i \left( \omega \frac{(N - l + 1)}{N + 1} + b \right) \right)^Y.$$

Here, $\tilde{g}(\omega_1, \omega_2)$ becomes:

$$\tilde{g}(\omega_1, \omega_2) = \exp \left[ -2CT(-Y)T(G^Y + M^Y) + f(\omega_1, \omega_2, 0, 0, 1, N) \right] \times \sum_{k=1}^{N} \exp \left( h(k) + f(\omega_1, \omega_1, -i, 0, -1, k) \left( \sum_{l=1}^{k-1} q(l) + \frac{1}{2} q(k) \right) \right),$$

Table 6.3: In this table our bounds for prices of arithmetic Asian call options with strike $K$ for the NIG model are shown. Our lower bound (LB) and upper bound (UB) are compared with the FM and Monte Carlo results (MC) of Ref. [184] and with the comonotonic upper bound (UBC). The parameter values used are: $S_0 = 100, T = 1, r = 0.0367, \alpha = 6.1882, \beta = -3.8941, \delta = 0.1622$. MR stands for the maximal relative error which is made when the average of the lower and the upper bound is used as an approximation. RLB stands for the relative error compared with the Monte Carlo result which is made when the lower bound is used as an approximation.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$K$</th>
<th>LB</th>
<th>FM</th>
<th>MC</th>
<th>UB</th>
<th>UBC</th>
<th>MR</th>
<th>RLB</th>
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<td>0.03%</td>
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<td>5.06060</td>
<td>5.06077</td>
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<td>0.03%</td>
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<td>1.01374</td>
<td>1.1865</td>
<td>1.3351</td>
<td>7.9%</td>
<td>0.08%</td>
</tr>
<tr>
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<td>90</td>
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<td>12.66118</td>
<td>12.66112</td>
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<td>13.067</td>
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<td>0.02%</td>
</tr>
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<td>1.03770</td>
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</tr>
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</tr>
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<td>0.03%</td>
</tr>
<tr>
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<td>n.a.</td>
<td>1.04446</td>
<td>1.21812</td>
<td>1.3811</td>
<td>7.7%</td>
<td>0.05%</td>
</tr>
</tbody>
</table>

Where $C$ lays down the overall level of activity, the parameters $G$ and $M$ determine the exponential decay on the right and the left of the Lévy density and $Y$ determines the fine structure of the stochastic process.
The comonotonic upper bound (UBC) stands for the maximal relative error which is made when the average of the lower and the upper bound is used as an approximation. RLB stands for the relative lower bound is used as an approximation.

Table 6.4: In this table our bounds for prices of arithmetic Asian call options with strike $K$ for the CGMY model are shown. Our lower bound (LB) and upper bound (UB) are compared with the FM and Monte Carlo results (MC) of Ref. [184] and with the comonotonic upper bound (UBC). The parameter values used are: $S_0 = 100$, $T = 1$, $r = 0.0367$, $C = 0.0244$, $G = 0.0765$, $M = 7.5515$ and $Y = 1.2945$. MR stands for the maximal relative error which is made when the average of the lower and the upper bound is used as an approximation. RLB stands for the relative error compared with the numerical quadrature result which is made when the lower bound is used as an approximation.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$K$</th>
<th>LB</th>
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<th>MC</th>
<th>UB</th>
<th>UBC</th>
<th>MR</th>
<th>RLB</th>
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</tr>
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<tr>
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<td>1.3162</td>
<td>1.4118</td>
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<td>0.06%</td>
</tr>
</tbody>
</table>

\[ h(k) = \bigg[ r - CT (-Y) \left( (M - 1)^Y + (G + 1)^Y - (G^Y + M^Y) \right) \bigg] \Delta k, \] (6.69)

\[ q(k) = \exp \left[ h(k) \right] \left( \exp \left[ f(\omega_1, \omega_1, -2i, -i, -1, k) \right] - \exp \left[ f(\omega_2, \omega_2, -i, 0, -1, k) \right] \right), \] (6.70)

\[ f(\omega_1, \omega_2, a, b, c, k) = CT (-Y) \Delta x \sum_{k=1}^{K} \left( \left( M - i \left( \omega_1 \frac{N-k+1}{N+1} + a \right) \right)^Y + \left( G + i \left( \omega_1 \frac{N-k+1}{N+1} + a \right) \right)^Y \right) \left( \left( M - i \left( \omega_2 \frac{N-k+1}{N+1} + b \right) \right)^Y + \left( G + i \left( \omega_2 \frac{N-k+1}{N+1} + b \right) \right)^Y \right). \] (6.71)

Also for the CGMY model Ref. [184] presents numerical quadrature and Monte Carlo results for arithmetic Asian option prices. For this model however Fusai et al. [184] claim that their Monte Carlo result is less reliable. Indeed we find that their Monte Carlo result is often below our lower bound. Therefore we compared with the numerical quadrature result to calculate the relative error made when the lower bound is chosen as an approximation. The comparison of our bounds with their results is shown in Table 6.4. The accuracy and the computational cost of the lower bound are practically the same as with the other models. For the upper bound though both the accuracy and the computational cost are less good as compared to the other models.

### 6.4.5 The Variance gamma Model

This model is not studied in Ref. [184], in Ref. [186] however several lower bounds for the price of a discrete arithmetic Asian call option are calculated. Yet Albrecher
et al. [186] use
\[ A = \frac{1}{N} \sum_{k=1}^{N} S_k \]
as definition for the discrete average, where we use expression (6.1) as definition.
For this model \( f(w) \) can be derived from results in Ref. [113]:
\[
 f(w) = -\frac{1}{v} \ln \left( 1 - i\theta vw + \frac{\sigma^2}{2} vw^2 \right),
\] (6.72)
where \( \sigma \) controls the volatility, \( \theta \) the skewness and \( v \) the kurtosis. The function \( g(\omega_1) \) becomes:
\[
g(\omega_1) = \exp \left\{ -\frac{\Delta}{v} \sum_{l=1}^{N} \Omega(\omega_1, l, 0) \right\} \left( \frac{S_0}{(N + 1)} \times \right.
\[
 \left\{ 1 + \sum_{k=1}^{N} \exp \left[ \left( \frac{rk + \frac{k}{v} \ln \left( 1 - \theta v - \frac{\sigma^2}{2} v \right)}{\Delta} \right) - K \right] \right\}
\] (6.73)
with this time
\[
\Omega(\omega, l, b) = \ln \left( 1 - i\theta v \left( \omega \frac{(N - l + 1)}{N + 1} + b \right) + \frac{\sigma^2}{2} v \left( \omega \frac{(N - l + 1)}{N + 1} + b \right)^2 \right).
\] (6.74)
For \( \tilde{g}(\omega_1, \omega_2) \) the result is:
\[
 \tilde{g}(\omega_1, \omega_2) = \exp \left[ f(\omega_1, \omega_2, 0, 0, 1, N) \right] \times \sum_{k=1}^{N} \exp \left( h(k) + f(\omega_1, \omega_1, -i, 0, -1, k) \right) \left( \sum_{l=1}^{k-1} q(l) + \frac{1}{2} q(k) \right),
\] (6.75)
with
\[
h(k) = \left[ r + \frac{1}{v} \ln \left( 1 - \theta v - \frac{\sigma^2}{2} v \right) \right] \Delta k,
\] (6.76)
\[
q(k) = \exp \left[ h(k) \right] \left( \exp \left[ f(\omega_1, \omega_1, -2i, -i, -1, k) \right] - \exp \left[ f(\omega_2, \omega_2, -i, 0, -1, k) \right] \right),
\] (6.77)
and
\[
f(\omega_1, \omega_2, a, b, c, k) = -\frac{\Delta}{v} \times \sum_{k_2=1}^{k} \left( \ln \left( 1 - i\theta v \left( \omega \frac{(N-k_2+1)}{N+1} + a \right) + \frac{\sigma^2}{2} v \left( \omega \frac{(N-k_2+1)}{N+1} + a \right)^2 \right) \right)
\] (6.78)
Table 6.5 demonstrates that also for this model our bounds perform very well. The computational cost for this model is comparable with the other models (with exception of the CGMY model for which the upper bound takes a longer time to evaluate).

80
### 6.5 Conclusions

Our lower bound is fast to evaluate, very accurate and it performs better than the existing lower bound for the VG model derived in Ref. [186]. For all the models except for the CGMY model the upper bound is still acceptably fast to evaluate although much slower than the lower bound; for the CGMY model the evaluation of the upper bound becomes quite slow. For all the examined models our upper bound is significantly more accurate than the comonotonic upper bound. Although it depends on the parameter values, for most relevant parameter values the upper bound is less accurate than the lower bound. Therefore we recommend using the lower bound as an approximation instead of the average of the lower and the upper bound.

As far as we know a lower bound has hitherto only been computed for the Black-Scholes model, whereas an upper bound was computed for the Black-Scholes model, the variance gamma, the normal inverse gaussian and the Meixner model (for the last three models see Ref. [244]). Our result extends previous work in the Black-Scholes setting to general Lévy processes, providing and improving analytical bounds for a variety of models.

Note that next to our bounds, other methods could be considered, based on previous results for other settings in the literature. We refer to Ref. [176] where an improved upper bound for Asian options was derived in a Black-Scholes framework, outperforming the bound obtained in Ref. [180]. An extension of this approach to general Lévy models could lead to a more accurate upper bound than the comono-

<table>
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<tr>
<th>$K$</th>
<th>LBA</th>
<th>LB</th>
<th>MC</th>
<th>error</th>
<th>UB</th>
<th>UBC</th>
<th>MR</th>
<th>RLB</th>
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<td>2.0%</td>
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</table>

Table 6.5: In this table our bounds for prices of arithmetic Asian call options for the VG model are shown. Our lower bound (LB) and upper bound (UB) are compared with the lower bound of Ref. [186] (LBA), a Monte Carlo simulation of $4 \times 10^8$ iterations and with the comonotonic upper bound (UBC). Next to the Monte Carlo simulation the error needed to find the 99% certainty interval is shown. The parameter values used are: $S_0 = 100$, $T = 10$, $N = 120$, $r = 0.03$, $\sigma = 0.2684$, $v = 1.1737$, $\theta = -0.1280$. MR stands for the maximal relative error which is made when the average of the lower and the upper bound is used as an approximation. RLB stands for the relative error compared with the Monte Carlo result which is made when the lower bound is used as an approximation.
tonic upper bound of [244]. We also refer to Ref. [245] where upper and lower bounds are derived for transition densities for rather general models. It is conceivable that an application of their approach to the pricing of Asian options could lead to similar bounds as presented in this chapter. An investigation of both alternatives, however, is beyond the scope of the present contribution.

6.A Derivation of the joint probability distribution $P(x_k, \bar{x})$

Here we derive the joint probability distribution $P(x_k, \bar{x})$ in terms of its characteristic function $\phi_{x_k,\bar{x}T}(\omega_1, \omega_2)$. This characteristic function is given by:

$$
\phi_{x_k,\bar{x}T}(\omega_1, \omega_2) = E \left[ \exp \left( i\omega_1 \bar{X}_T + i\omega_2 X_k \right) \right]
= E \left[ \exp \left( i\omega_1 \frac{1}{N+1} \sum_{l=1}^{N} X_{t_l} + i\omega_2 \sum_{l=1}^{k} X_{t_l}^{\Delta} \right) \right]
= E \left[ \exp \left( i\omega_1 \frac{1}{N+1} \sum_{l=1}^{N} (N-l+1) X_{t_l}^{\Delta} + i\omega_2 \sum_{l=1}^{k} X_{t_l}^{\Delta} \right) \right]
= E \left[ \exp \left( i\omega_1 \frac{1}{N+1} \sum_{l=k+1}^{N} (N-l+1) X_{t_l}^{\Delta} \right) \right]
= E \left[ \prod_{l=1}^{k} \exp \left( i \left( \frac{(N-l+1)}{N+1} + \omega_2 \right) X_{t_l}^{\Delta} \right) \prod_{l=k+1}^{N} \exp \left( i \omega_1 \frac{(N-l+1)}{N+1} X_{t_l}^{\Delta} \right) \right].
$$

(6.79)

Because of the independency of the increments, the order of the expectation value and the multiplications can be changed. We then recognize the characteristic function of the increments. The characteristic function $\phi_{x_k,\bar{x}T}(\omega_1, \omega_2)$ then becomes:

$$
\phi_{x_k,\bar{x}T}(\omega_1, \omega_2) = \prod_{l=1}^{k} \phi_{X_{t_l}^{\Delta}} \left( \frac{(N-l+1)}{N+1} + \omega_2 \right) \prod_{l=k+1}^{N} \phi_{X_{t_l}^{\Delta}} \left( \frac{(N-l+1)}{N+1} \right)
= \exp \left[ \left( \sum_{l=1}^{k} f \left( \frac{(N-l+1)}{N+1} + \omega_2 \right) + \sum_{l=k+1}^{N} f \left( \frac{(N-l+1)}{N+1} \right) \right) \Delta \right].
$$

(6.80)

The probability distribution can then be recovered from the characteristic function:

$$
P(\bar{x}, x_k) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp \left( -i\omega_1 \bar{x} - i\omega_2 x_k \right) \phi_{x_k,\bar{x}T}(\omega_1, \omega_2) \, d\omega_1 \, d\omega_2.
$$

(6.82)
6.B Derivation of formula (6.24)

Here expression (6.23) for the lower bound of $C$ is simplified. Remember that expression (6.23) was given by:

$$C \geq e^{-rT} \int_a^\infty \left[ \frac{S_0}{N+1} \left( 1 + \sum_{k=1}^N e^{(r-m)k\Delta} E \left[ e^{X_k|X_T = \bar{x}} \right] dt \right) - K \right] P_{X_T}(\bar{x}) \, d\bar{x}. \quad (6.83)$$

Substituting formula (6.21) for $E \left[ \exp (X_k) | X_T = \bar{x} \right]$ this becomes:

$$C \geq e^{-rT} \int_a^\infty d\bar{x} P_{X_T}(\bar{x}) \times$$

$$\left\{ \frac{S_0}{(N + 1)} \left[ 1 + \frac{1}{2\pi P(\bar{x})} \sum_{k=1}^N e^{(r-m)k\Delta} \int_{-\infty}^{+\infty} e^{-i\omega_1 \bar{x}} \phi_{X_k,X_T}(\omega_1, -i) \, d\omega_1 \right] - K \right\}. \quad (6.84)$$

Bringing $P(\bar{x})$ inside the brackets and filling in equation (6.16) for $P(\bar{x})$ one obtains

$$C \geq e^{-rT} \int_a^\infty \left[ \frac{S_0}{(N+1)} \left( \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega_1 \bar{x}} \phi_{X_T}(\omega_1) \, d\omega_1 + \frac{1}{2\pi} \sum_{k=1}^N e^{(r-m)k\Delta} \int_{-\infty}^{+\infty} e^{-i\omega_1 \bar{x}} \phi_{X_k,X_T}(\omega_1) \, d\omega_1 \right) \right] d\bar{x},$$

$$= \frac{e^{-rT}}{2\pi} \int_{-\infty}^{+\infty} d\bar{x} \theta(\bar{x} - a) \int_{-\infty}^{+\infty} d\omega_1 e^{-i\omega_1 \bar{x} \times}$$

$$\left[ \frac{S_0}{(N+1)} \left( \phi_{X_T}(\omega_1) + \sum_{k=1}^N e^{(r-m)k\Delta} \phi_{X_k,X_T}(\omega_1, -i) \right) - K \phi_{X_T}(\omega_1) \right]. \quad (6.85)$$

Introducing the notation

$$g(\omega_1) = \frac{S_0}{(N+1)} \left( \phi_{X_T}(\omega_1) + \sum_{k=1}^N e^{(r-m)k\Delta} \phi_{X_k,X_T}(\omega_1, -i) \right) - K \phi_{X_T}(\omega_1) \quad (6.86)$$

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and the Fourier representation of the Heaviside step function $\theta(.)$, expression (6.85) becomes:

$$
\frac{e^{-rT}}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{i}{q + i\eta} \exp(-iq(\bar{x} - a)) \frac{dq}{2\pi} \int_{-\infty}^{+\infty} \exp(-i\omega_1 \bar{x}) g(\omega_1) d\omega_1 d\bar{x}
$$

$$
= \frac{e^{-rT}}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{i}{q + i\eta} \exp(iqa) \exp(-i(\omega_1 + q) \bar{x}) \bar{x} g(\omega_1) \frac{dq}{2\pi} d\omega_1
$$

$$
= \frac{e^{-rT}}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{i}{q + i\eta} \exp(iqa) \delta(\omega_1 + q) g(\omega_1) dq d\omega_1
$$

$$
= \frac{e^{-rT}}{2\pi} \int_{-\infty}^{+\infty} \frac{i}{-\omega_1 + i\eta} \exp(-i\omega_1 a) g(\omega_1) d\omega_1.
$$

(6.87)

For the numerical integration it is convenient to write this as:

$$
\exp(-rT) \left[ \frac{g(0)}{2} - \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{\exp(-i\omega_1 a) g(\omega_1) - g(0)}{\omega_1} d\omega_1 \right],
$$

(6.88)

which is expression (6.24).

### 6.C Derivation of expression 6.27

The derivation starts by introducing the following notation:

$$
G = \exp\left( \frac{1}{N+1} \left( \sum_{k=0}^{N} \ln(S_k) \right) \right)
$$

$$
= \exp\left( \frac{1}{N+1} \left( \ln(S_0) + \sum_{k=1}^{N} \ln\{S_0 \exp\{(r - m) \Delta k + X_{t_k}\}\} \right) \right)
$$

$$
= S_0 \exp\left( \frac{1}{N+1} \sum_{k=1}^{N} X_{t_k} + \frac{(r - m)}{2} T \right)
$$

$$
= S_0 \exp\left( \bar{X} + \frac{(r - m)}{2} T \right)
$$

Before we proceed it is useful to first prove the following lemma

**Lemma 9** $A \geq G$ (where $A$ is the arithmetically averaged asset price, expression (6.1)).
Proof. Notice that the following equality holds

\[
G = \exp \left( \ln \left( \prod_{k=0}^{N} S_k \right)^{\frac{1}{N+1}} \right)
= \left( \prod_{k=0}^{N} S_k \right)^{\frac{1}{N+1}}
\]

The maximum value of \( G \) supposing that \( A \) equals \( a \) can be found with the help of a Lagrange multiplier. If

\[
f (S_0, \ldots, S_N, \lambda) = \prod_{k=0}^{N} S_k + \lambda \left( \frac{1}{N+1} \sum_{k=0}^{N} S_k - a \right)
\]

Then the \( S_k \) for which \( G \) attains its maximum value and \( A = a \) are a solution of the following system of equations.

\[
\begin{align*}
S_1S_2\ldots S_N + \frac{\lambda}{N+1} &= 0 \quad (6.89a) \\
S_0S_2\ldots S_N + \frac{\lambda}{N+1} &= 0 \quad (6.89b) \\
\vdots & \\
S_0S_1\ldots S_{N-1} + \frac{\lambda}{N+1} &= 0 \quad (6.89c) \\
\frac{1}{N+1} \sum_{k=0}^{N} S_k - a &= 0 \quad (6.89d)
\end{align*}
\]

From equations (6.89a)-(6.89c) it follows that:

\[
\Rightarrow S_1S_2\ldots S_N = S_0S_2\ldots S_N = \cdots = S_0S_1\ldots S_{N-1}
\Rightarrow \frac{1}{S_0} = \frac{1}{S_1} = \cdots = \frac{1}{S_N}
\Rightarrow S_0 = S_1 = \cdots = S_N
\]

From this and expression (6.89d) it follows that:

\[
S_0 = S_1 = \cdots = S_N = a.
\]

Therefore the maximum value of \( G \) if \( A \) equals \( a \) is \( a \) and we conclude that \( A \geq G \).

Now we can proceed with the derivation of the upper boundary for

\[
E \left[ (A - K)_+ \right] - E \left[ (E \left[ A - K \mid \bar{X} \right])_+ \right].
\]

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Since $G$ is a continuous function of $\bar{X}$, $G$ and $\bar{X}$ produce the same sigma algebra and the conditional expectation on $\bar{X}$ is the same as the conditional expectation on $G$, and the previous equation becomes:

$$
E \left[ (A - K)_+ \mid G \right] - E \left[ (E [A - K] \mid G)_+ \right] 
= E \left[ (A - K)_+ \mid G \right] 1_{G<K} + E \left[ (A - K)_+ \mid G \right] 1_{G \geq K} 
- E \left[ (E [A - K] \mid G)_+ 1_{G<K} + (E [A - K] \mid G)_+ 1_{G \geq K} \right].
$$

Since $A \geq G$ the previous expression becomes:

$$
E \left[ (A - K)_+ \mid G \right] 1_{G<K} + E [A - K] 1_{G \geq K} 
- E \left[ (E [A - K] \mid G)_+ 1_{G<K} + (E [A - K] \mid G)_+ 1_{G \geq K} \right].
$$

Again we use the fact that the conditional expectation on $\bar{X}$ is the same as the conditional expectation on $G$.

$$
E \left[ (A - K)_+ \mid \bar{X} \right] - (E [A - K] \mid \bar{X})_+ 1_{\bar{X} < b} 
= E \left[ (A - K)_+ \mid \bar{X} \right] - (E [A - K] \mid \bar{X})_+ 1_{\bar{X} < b} \right] 
= \frac{1}{2} \left( E \left[ (U - E (U \mid \bar{X}) \mid \bar{X} \right] - \frac{m-r}{2} T \right)
$$

with

$$
b = \ln \left( \frac{K}{S_0} \right) + \frac{m-r}{2} T
$$

If $A - K$ is denoted by $U$ then $E \left[ (A - K)_+ \mid \bar{X} \right] - (E [A - K] \mid \bar{X})_+$ becomes:

$$
E (U_+ \mid \bar{X}) - E (U_+ \mid \bar{X})_+ \leq \frac{1}{2} \left( E \left[ (U - E (U \mid \bar{X}) \mid \bar{X} \right] - \frac{m-r}{2} T \right)
$$

this turns expression (6.90) in to

$$
\frac{1}{2} E \left[ \sqrt{V ar \left[ \frac{1}{N+1} \sum_{k=0}^{N} S_k - K \mid \bar{X} \right]} 1_{\bar{X} < b} \right] ,
$$

which was to be shown.
Chapter 7

Option implied density

In this chapter we will focus on the determination of the asset price probability density underlying observed option prices. This is in some way the inverse problem to what we have been doing in the previous chapters. Until now, when we wanted to calculate European vanilla options for a range of strikes \( K \) with a certain maturity \( T \), we tried to determine the asset price distribution \( P(S_T) \) for a certain model, and once \( P(S_T) \) was found the option could be calculated. Now, we observe option prices for strikes \( K \) with maturity \( T \), and from this we want to derive the \( P(S_T) \) that the set of market participants generated to produce these prices. There are a lot of approaches that can be followed. One can assume a functional form for the underlying distribution and determine its parameters doing a least squares fit to the observed prices. A popular choice for this functional form is a combination of lognormals \([246–250]\). One can also use polynomials to take into account deviations from the lognormal distribution \([251–253]\).

Another approach is to use the relation first put forward by \([254]\) that the implied distribution can be obtained by taking the second derivative of the option price with respect to the strike price. The problem here is that we only have option prices for a discrete set of strikes. Furthermore option data often (always) contains errors, and the second derivative is extremely sensitive to such errors. One way to cope with these problems is to use the Black-Scholes pricing formula to translate the prices as a function of strikes to volatilities as a function of strikes. Then some function is fitted to these volatilities. This function is then mapped back to prices. The second derivative of this function then gives the implied density function \([248, 250, 255–257]\). For a more thorough discussion on the possible methods to infer the implied density we refer to \([246, 247, 258–263]\).

A large part of the literature uses one of these methods to extract information from market data without investigating the performance of the methods itself. Another part uses market data to test the performance of the methods for example by using several methods and comparing the different results \([258]\). Another example is found in Ref. \([259]\) where the goal is to discriminate between a double lognormal based method and an implied volatility smoothing based method on grounds of stability with respect to small errors. This is done by first estimating the implied pdf of real data using both methods, then small errors are added to the option prices, and the implied density is estimated again.
Only a small part of the literature uses simulated data to test the performance of the methods used to derive the implied density. Ref. [264] uses simulated data to study the influence of the incomplete set of strikes and the presence of a bid-ask spread. Ref. [247] might be closest to what we want to do. The paper tests two approaches to estimate the implied density function; the method based on interpolating the implied volatility smile and the method based on a mixture of two lognormals. The ability to recover a distribution simulated with Heston’s stochastic volatility model is investigated. The stability and the robustness to small errors is investigated as well as the ability to recover the statistics from the true distribution.

In this chapter we test methods for determining the implied density. More precisely we want to investigate the robustness of three methods against errors and with respect to data coming from different models. The three methods we will inquire into for determining the implied density are the double lognormal method, the method based on smoothing the implied volatility surface and a method we constructed ourselves. This method is more elaborate than the others and is not meant for pricing purposes. The goal is to have a method which can be used to answer questions such as: Is the interest rate implied by option prices comparable to the interest rate implied by bond prices? Is the drift implied by option prices more or less the same as the drift implied by forward prices? To what extent are the characteristics of the distribution implied by historical data similar to those implied by option prices?

7.1 Setting the stage

7.1.1 The double lognormal approach

A commonly used and straightforward method is the “double lognormal approach”. For this approach one assumes that the underlying density function of \( S_T \) is given by a double lognormal.

\[
P(S_T) = \frac{b}{S_T s_1 \sqrt{2\pi T}} \exp \left( -\frac{\left( \ln (S_T) - \ln (S_0) - \left( m_1 - \frac{s_1^2}{2} \right) T \right)^2}{2s_1^2 T} \right) \\
+ \frac{1 - b}{S_T s_2 \sqrt{2\pi}} \exp \left( -\frac{\left( \ln (S_T) - \ln (S_0) - \left( m_2 - \frac{s_2^2}{2} \right) T \right)^2}{2s_2^2 T} \right),
\]

where \( m_1, m_2 \) are drifts, \( s_1, s_2 \) volatilities and \( b \) determines the relative contribution of the two lognormal densities. The price of a call option is then given by:

\[
C = be^{-rT} \left[ e^{m_1 T S_0 N (d_1)} - KN (d_2) \right] + (1 - b) e^{-rT} \left[ e^{m_2 T S_0 N (d_3)} - KN (d_4) \right]
\]

(7.1)
with
\[ d_1 = \frac{\ln \left( \frac{S_0}{K} + \left( m_1 + \frac{s_1^2}{2} \right) T \right)}{s_1 \sqrt{T}} \]
\[ d_2 = \frac{\ln \left( \frac{S_0}{K} + \left( m_1 - \frac{s_1^2}{2} \right) T \right)}{s_1 \sqrt{T}} \]
\[ d_3 = \frac{\ln \left( \frac{S_0}{K} + \left( m_2 + \frac{s_2^2}{2} \right) T \right)}{s_2 \sqrt{T}} \]
\[ d_4 = \frac{\ln \left( \frac{S_0}{K} + \left( m_2 - \frac{s_2^2}{2} \right) T \right)}{s_2 \sqrt{T}} \]

The parameters \( s_1, m_1, s_2, m_2, b, r \) are then determined by minimizing the least square distance between the real price and the price given by expression (7.1). Since this method has six fitting parameters, \( s_1, m_1, s_2, m_2, b, r \) it can strand in a local minimum and give unreliable results.

7.1.2 The implied volatility surface approach

For the next two methods let us first recapitulate how a probability density function can be determined from option price data. Suppose that at a certain time \( T \) asset prices have a probability distribution \( p(S_T, T|S_0) \). Then, call prices can be calculated as follows:
\[ C(S_0, K, T) = e^{-rT} \int_K^\infty (S_T - K) p(S_T, T|S_0) dS_T \]

Where \( e^{-rT} \) is a discount factor. If this formula is differentiated twice with respect to \( K \) we get:
\[ \frac{\partial^2 C(S_0, K, T)}{\partial K^2} \bigg|_{K=S_T} = e^{-rT} \left[ \frac{\partial}{\partial K} \left( - \int_K^\infty p(S_T, T|S_0) dS_T \right) \right] \bigg|_{K=S_T} \]
\[ = e^{-rT} p(K, T|S_0) \bigg|_{K=S_T} = e^{-rT} p(S_T, T|S_0) \]

The interest rate can be determined because the integral of a distribution function is equal to one. If this formula is integrated we get:
\[ \int_{-\infty}^\infty \frac{\partial^2 C(S_0, K, T)}{\partial K^2} dK = e^{-rT} \]

The implied probability distribution is then given by:
\[ p(S_T, T|S_0) = e^{rT} \frac{\partial^2 C(S_0, K, T)}{\partial K^2} \bigg|_{K=S_T} \quad (7.2) \]

As already mentioned there are some problems to bring this theoretical relation in to practice (see Fig. 7.1). Problems such as a discrete set of strikes or errors. For example, option prices are available only for a discrete set of strike values. Also, the market mechanism of bidding and asking results in a “measurement error” or uncertainty on the observed option prices.
Figure 7.1: The left panel shows S&P 500 call option prices as a function of strike. The settlement date is 1/7/09 and the maturity date 21/8/09. The right panel shows the corresponding implied volatility curve.

A popular approach to cope with these problems is the one based on smoothing the volatility smile. For this method option prices are first transformed to a certain volatility curve. If the interest rate \( r \) and the drift \( \mu \) can not be determined by other methods one has to assume certain values for them, we will work with \( r = 0.05 \) and \( \mu = 0.05 \). In a Black-Scholes setting the call option price is given by:

\[
C = e^{-rT} \left[ e^{\mu T} S_0 N(d_1) - K N(d_2) \right]
\]

with

\[
d_1 = \frac{\ln \left( \frac{S_0}{K} + \left( \mu + \frac{s^2}{2} \right) T \right)}{s_1 \sqrt{T}} \quad d_2 = \frac{\ln \left( \frac{S_0}{K} + \left( \mu - \frac{s^2}{2} \right) T \right)}{s_1 \sqrt{T}}
\]

As a first step we use this formula to go from a vector of call prices as a function of strike to a vector of volatilities as a function of strike. Next this vector of volatilities is smoothed with a cubic spline. This smoothed volatility vector is then mapped again on to a vector of call prices using expression (7.3). Taking the second derivative of these call prices gives the density function according to expression (7.2). This method will be referred to as the implied volatility surface (IVS) method and the corresponding implied density as the IVS implied density.

7.1.3 Smoothing the option prices directly

The last method is a method we constructed ourselves. Our goal is to draw a smoothing function through the option price so we can take the second derivative. We know that this second derivative is always positive and should not contain too many bumps and spikes. Instead to try to impose all these conditions on a polynomial fit we tried to do a piecewise fit with a function which already approximates the desired behavior. This desired behavior is a monotonically decreasing function which approximates a linear decreasing function in the left limit and a constant function in the right limit. An easy function with this behavior is:

\[
\frac{a}{x - b + \sqrt{(x - b)^2 + c}} + d. \quad (7.4)
\]
The second derivative of this function is

\[ \frac{a}{(c + (b - x)^2)^{3/2}} \]

Another obvious choice for the interpolating function is the Black-Scholes formula. The algorithm we propose goes as follows:

- The fitting algorithm is started in the region around the forward price. Since the option is mostly traded around the forward price (the center) it is fair to assume that the data is most reliable there.

- It is not obvious how many points one should try to fit at the same time. Taking too much points will make the fit worse, taking too few points will make the contribution of “outliers” too big. Therefore the fitting interval is taken proportional to the variance. This variance is calculated assuming a lognormal model with some reasonable parameters. Maybe because these parameters are too far from the real ones or because the lognormal distribution is too far from the real distribution this didn’t always give a good result. Sometimes it was better to take the number of fitting points proportional to the total number of points. We added an extra parameter to control the amount of fitting points.

- After the center, the left and the right tails are fitted. To determine an extra point in the left tail the fitting interval is shifted one place to the left. So each point in the left tail is fitted together with points on its right side. The reason for this is again that the closer a point is to the center the more reliable it is presumed to be. The right tail is treated similarly.

- To improve the smoothness of the density function we add an extra term to the sum of the squares in the minimization procedure. If \( g \) is the fitting function with parameter vector \( q \), \( pv \) the vector of option prices and \( \mathbf{Kv} \) the vector of strikes then

\[ \sum \left( \frac{g(q, \mathbf{Kv}) - pv}{pv} \right)^2 + e \left( \frac{\partial^2 g|_{q_0, Kv(1)}}{\partial K^2} - \frac{\partial^2 g|_{q_0, Kv(1)}}{\partial K^2} \right)^2 \]  

needs to be minimized as a function of the parameter vector \( q \). We use the Black-Scholes price and expression (7.4) as fitting function. When an extra point is fitted we also calculate, in this point, the second derivative corresponding to the present parameter set and the second derivative corresponding to the parameter set \( a_0 \) from the previous fit. The extra term is proportional to the squared difference between these two second derivatives. There is than an extra parameter \( e \) which controls the importance of this extra term. This parameter can be different for the right and the left tail. Expression (7.5) applies to the left tail since the extra point that is fitted corresponds to \( Kv(1) \).

- With every step the least squares error from the fit using (7.4) is compared with the least squares error resulting from using the Black-Scholes price as fitting function. The fit with the smallest least squares error is used.
Finally a large amount of fits are generated for various values of the parameters which control the number of fitting points and the smoothness of the tails. From all these fits we remove the fits which have to many spikes and bumps. An obvious way to proceed is to select out of the remaining fits select the one with the smallest overall least square error, we will refer to the distribution implied by this method as the least squares LS implied distribution. Though we noticed that often better fits or fits that were better in a certain region than the "least squares fit" passed the selection. Therefore we also calculate the average of all the fits passing the selection procedure. we will refer to the distribution implied by this method as the average AV implied distribution

7.2 Simulating data

Figure 7.2: Here the distributions to test the methods from the previous section are pictured. From left to right time ranges from 0.0384 to 0.5 to 1.5. The full red curve represents the lognormal distribution, the blue dots the Heston distribution and the blue crosses the CGMY distribution. Parameter values are given in the text and $S_0 = 925$.

Rather than using market data, with an essentially still uncertain probability distribution function, we use simulated data based on known probability distribution functions, and add noise. This allows us to accurately test and benchmark the different approaches to imply a PDF from noisy data. Three different models are used to test the methods of the previous section. The first model is of course the Black-Scholes model, with as interest rate $r = 0.03$, drift $\mu = 0.05$, volatility $\sigma = 0.2$. The second model, also not a surprise, is the Heston model. The parameters for the Heston model are chosen so that the resulting distribution lies close to the lognormal one, the mean reversion speed $\kappa = 2$, the mean reversion level $\theta = 0.04$, the volatility of the volatility $\sigma = 0.1$, the correlation $\rho = 0.5$, the variance at inception $v_0 = 0.0437$, see Fig. 7.2. Then it can be checked whether a small deviation from the lognormal model already influences the performance of the models. The third model is the CGMY model with parameter values: $C = 0.0244$, $G = 0.0765$, $M = 7.5515$, $Y = 1.2945$. The corresponding distribution differs substantially from the lognormal distribution as can be seen in figure 7.2. Nevertheless the parameter values for this model are realistic since they were obtained by Ref. [105] by calibrating
on a set of European call options on the S&P500 index. For these models prices and distributions are calculated for a time to maturity of 0.0384, 0.5 and 1.5 year.

Figure 7.3: This figure is an illustration of the relative errors for different values of $\eta$. Each panel represents a different time to maturity. The first panel has a time to maturity of 0.0384 years, the middle one has a maturity of 0.5 year and the right one a maturity of 1.5 year. The red dots represent an $\eta$ value of 1, the blue plus signs an $\eta$ value of 10, the green crosses an $\eta$ value of 50 and the black asterisks an $\eta$ value of 100.

After the prices corresponding to these models are calculated noise is added to the prices. The prices plus noise should be similar to market data. Different choices can be made for replicating option market data. One can choose to work with last trade data. The benefit of doing this is that there is one realistic price per strike. The downside is that data for different strikes can originate from trades at different times and theoretically when comparing prices originating from different times you should all discount them to the same time. Another possibility is to use bid-ask price data, than you have a vector of prices which is relevant at the same time. The problem there is that people can state unrealistic high ask prices, ask prices at which there will never be traded. The same can be said about bid prices. In the following we will try to replicate last trade data. There are a lot of assets for which options are traded actively enough for last trade data to be useful. We want to construct the noise in such a way that it replicates the fact that the noise of financial data is smallest around the forward value $F$ (where $F = S_0 \exp(\mu T)$ which for the present values of $T$ results in the array (926.776; 948.42; 997.04) for $F$) and largest in the tails. To determine where the tails start we use the standard deviation $s$. For each option price a relative error is chosen from a uniform distribution on an interval $[-\beta, \beta]$ where $\beta$ is determined by:

$$\beta(K) = \eta \left( \frac{0.00025 |F - K|}{s} + 0.0001 \right)$$

If $\eta$ is 1 the relative error ranges from 0.0001 per unit in the centre to 0.0011 per unit in the tails (four standard deviations away from the centre). If $\eta$ is 100 the relative error ranges from 0.01 per unit in the centre to 0.11 per unit in the tails. Figure 7.3 illustrates the relative errors of the simulated data where the Black-Scholes price of the previous section was used as the price without errors.
7.3 Results and discussion

The results for the implied density functions obtained by the different methods for the maturity times and values of $\eta$ of the previous section are presented in figures 7.4 to 7.9. The performance of the different methods is summarized in table 7.1. In this table a method which renders a better implied distribution has more plus signs, if the implied distribution is unacceptable a minus sign is assigned to it. We discriminate between good and bad fits in the following way. If $d_i$ are the data points from the original distribution and $d'_i$ are the data points of the implied distribution then we calculate a normalized average error $ne$.

$$ne = \frac{1}{N \max(d_i)} \sum_i |d_i - d'_i|$$

Absolute errors are used instead of relative errors since we want to concentrate on the center of the distribution. To be able to compare the average errors of different implied distributions the average error is divided by the maximum value of the original distribution. For example if $ne = 1$ then, on average, the absolute error in each point is as large as the maximum value of the distribution, and the corresponding implied distribution is obviously worthless. If the implied distribution is zero in every data point then $ne$ lies close to 0.5. When $ne \leq 0.1$ the implied distribution starts to look like the original distribution, and we assign a $-$ sign to the distributions for which $ne > 0.1$. If $0.02 \leq ne \leq 0.1$ a $+$ sign is assigned to the implied density, when $0.004 \leq ne \leq 0.02$ a $++$ sign and when $ne \leq 0.004$ a $++++$ sign.

We conclude that the double lognormal method is least influenced by errors. On the other hand this method is unstable as a consequence of being based on a six dimensional minimization routine. This is a problem which is not encountered extremely in literature because it is very often assumed that the interest rate and the drift can be determined by other means (bond and future prices). Two parameters can then be eliminated and the minimization routine becomes much more stable. The double lognormal, entirely as expected, is least flexible concerning the reproduction of models differing substantially from the lognormal one. The method based on smoothing the volatility smile seems to be flexible as far as it concerns handling data coming from models differing substantially from the lognormal one. Nevertheless also for this model the data can be too far from the lognormal one to be tractable. This method is extremely sensitive to random errors. The method might also profit from the knowledge of the interest rate and the drift. As far as the method we constructed ourselves concerns, this method is more suited to match data from models differing substantially from the double lognormal method. The method is also much more robust with respect to errors than the method based on smoothing the volatility smile. Clearly more work needs to be done before this method can be adapted to a satisfying method. It is doubtful whether it is possible to construct a method to determine the implied distribution from option prices alone, without using information about the interest rate and the drift of the underlying process. Therefore it might be more appropriate to try to adapt our method so it can take such information into account.
Figure 7.4: In this figure results for the Black-Scholes data are pictured. The left column are results corresponding to $a = 1$, for the right column $a = 10$. Going from the top row to the bottom row the time $T$ to maturity takes the values 0.038, 0.5 and 1.5. The red line is the original “real” distribution. The blue dotted line is the result from our algorithm when we take the distribution whose corresponding price vector has the least square error with the real prices. the blue x-es result from taking the average of all the density vectors passing our selection procedure. The green plus signs are the result from the double lognormal approach and the black asterisks are the result from the implied volatility smoothing approach.
Figure 7.5: This figure shows the implied densities corresponding to the Black-Scholes data with errors $a = 50$ (left column) and $a = 100$ (right column). Again time increases from the top row to the bottom one, the red line is the original distribution, the green plus signs represent the distributions implied by the lognormal approach and the blue dots represent the LS implied distribution and the blue x-es represent the AV implied distribution. The distributions implied by the IVS method are omitted because they obscure the other distributions.
Figure 7.6: In this figure results for the Heston data are pictured. The Heston parameters are given in the main text. The left column are distributions corresponding to prices without error, in the right column $a = 1$. Also time increases from the top row to the bottom one, the red line is the original distribution, the green plus signs represent the distributions implied by the lognormal approach and the blue dots represent the LS implied distribution, the blue x-es represent the AV implied distribution and the distributions implied by the IVS method are represented by black asterisks.
Figure 7.7: This figure shows the implied densities corresponding to the Heston data with errors $a = 10$ (left column) and $a = 50$ (right column). Again time increases from the top row to the bottom one, the red line is the original distribution, the green plus signs represent the distributions implied by the lognormal approach and the blue dots represent the LS implied distribution, the blue x-es represent the AV implied distribution and the distributions implied by the IVS method are represented by black asterisks (omitted when too messy).
Figure 7.8: In this figure results for the CGMY data are pictured. The left column are distributions corresponding to prices without error, in the right column $a = 1$. Also time increases from the top row to the bottom one, the red line is the original distribution, the green plus signs represent the distributions implied by the lognormal approach and the blue dots represent the LS implied distribution, the blue x-es represent the AV implied distribution and the distributions implied by the IVS method are represented by black asterisks (omitted when too messy).
Figure 7.9: This figure shows the implied densities corresponding to the Heston data with errors $a = 10$ (left column) and $a = 50$ (right column). Again time increases from the top row to the bottom one, the red line is the original distribution, the green plus signs represent the distributions implied by the lognormal approach and the blue dots represent the LS implied distribution, the blue x-es represent the AV implied distribution and the distributions implied by the IVS method are represented by black asterisks (omitted when too messy).
Table 7.1: Summary of the performance of the different methods to derive the implied distribution function. The same underlying models, maturity times and values for $\eta$ are used as in the previous section. The double lognormal method is abbreviated by dl and the method based on smoothing the volatility surface by sv.

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Chapter 8

Conclusions

In chapter 3 we have developed a path-integral method to derive closed-form analytical formulas for the asset price distribution in the Heston stochastic volatility model. Closed-form formulas are obtained for the vanilla option price. The presented results correspond to the known semi-analytic results obtained from solving the partial differential equation [10] by standard techniques. The flexibility of the approach is demonstrated by extending the results to the case where the interest rate is a stochastic variable as well, and follows a CIR process. For this case, to the best of our knowledge, no exact analytical solutions have been derived before. For the very similar problem, where the interest rate is modeled by the Hull-White model, an analytic formula is derived by ref. [187]. We have checked our semi-analytical results for the model with both stochastic volatility and stochastic interest rate against a Monte-Carlo simulation. The quantitative analysis shows that the effect of stochastic interest rate on the Heston model can be as large as the effect of the stochastic volatility on the Black-Scholes model. Finally, the analogy between stochastic interest rate models and path dependent options makes our method promising for the pricing of some exotic derivative products.

In chapter 4 we derived a closed-form pricing formula for an average price as well as an average strike geometric Asian option within the path integral framework. The result for the average price Asian option corresponds to that found by Linetsky [159]. The result for the average strike Asian option was compared to a Monte Carlo simulation. We found that the agreement between the numerical simulation and the analytical result for an average strike Asian option is such that they coincide to within a relative error of less than 0.3 % for at least 500000 samples and 100 timesteps.

In chapter 5 we presented a method which makes it possible to extend the propagator for a general stochastic volatility model to the propagator of that stochastic volatility model including an arbitrary jump process in the asset price evolution. This procedure, applied to the Heston model, leads to similar results as those obtained in Ref. [188], which gives us confidence in the present treatment. The stationary volatility distribution of the Heston model, however, does not correspond to the observed lognormal distribution [5, 36, 233] in the market. The exponential Vašíček model does have the lognormal distribution as its stationary distribution. Therefore we used this model for the volatility to illustrate the method presented in section 5.1.
For this model no closed form pricing formulas for the propagator or vanilla option prices exist. We first derive approximative formulas for the propagator and vanilla option prices for this model without jumps, using path integral methods. Then, we extended this result to the case where the asset price evolution contains jumps. Both results were checked with a Monte Carlo simulation, providing a parameter range for which the approximation is valid. For the jump free situation we specified a parameter range for which our pricing formulas are accurate to within 3%. They become more accurate in the limit $\frac{\beta}{\gamma^2} >> 1$ where $\beta$ is the mean reversion rate and $\gamma$ is the volatility of the volatility. For the situation where the model contains jumps the Monte Carlo simulation suggest that the approximation error is typically less than 2%.

In chapter 6 we derived bounds for the price of a discretely monitored arithmetic Asian option when the underlying asset is assumed to follow a general Lévy process. We tested these bounds for Kou’s model, Merton’s model, the normal inverse Gaussian model, the CGMY model and the variance gamma model. Additionally we evaluate the comonotonic upper bound for all these models. As far as we know this bound has hitherto only been computed within the Black-Scholes framework. This numerical analysis leads us to the following observations. The lower bound is fast to evaluate and very accurate. Furthermore our lower bound performs better than the existing lower bound for the variance gamma model derived in Ref. [186]. For all the models except for the CGMY model the upper bound is still acceptably fast to evaluate although much slower than the lower bound. For the CGMY model the evaluation of the upper bound becomes quite slow with respect to the other models. For all the examined models our upper bound is significantly more accurate than the comonotonic upper bound. Although it depends on the parameter values, for most relevant parameter values the upper bound is less accurate than the lower bound. Therefore we recommend using the lower bound as an approximation instead of the average of the lower and the upper bound. Our results extend previous work in the Black-Scholes setting to general Lévy processes, providing and improving analytical bounds for a variety of models.

The main idea behind chapter 7 is to inquire into the information content of option price data. However, in our opinion, the existing methods for extracting information are not tested rigorously enough to be reliable. Therefore we tested two of the popular methods, namely the double lognormal approach and implied volatility smoothing, ourselves. Since preliminary investigations already suggested that these methods performed poorly in some situations, we suggested our own method. Then we tested this method together with the double lognormal approach and the approach based on smoothing the implied volatility surface. We tested the stability of these methods with respect to data coming from different models and with respect to random errors. The approach based on smoothing the implied volatility surface is very sensitive to random errors, but it is able to reproduce the implied density corresponding to models differing from the lognormal one. For the double lognormal approach the opposite is true, it is stable with respect to random errors but it performs poorly when it comes to reproducing densities coming from models differing from the lognormal one. Our own method seems to perform better than the other two but is also clearly not a reliable method either. Although we will
try to improve our method we doubt whether it is possible to determine the implied volatility without using other information. For example information coming from observed forward and bond prices.

In summary, we have investigated different aspects of option pricing. Firstly novel exact expressions are derived for the Heston model with stochastic interest rate, and for the geometric Asian option. Secondly we present a methodology to extend solutions for a stochastic volatility model to solutions for this model including jumps. This is applied to the exponential Vašíček model with Kou and Merton jump diffusions. Thirdly we derive pricing bounds for discrete arithmetic asian options under lévy models. Finally we investigate the “reverse engineering” problem of finding the implied probability density function for the underlying asset, from option prices.

8.1 Publications related to the results obtained in the thesis


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