Abstract

It is well-known that uncovering spectral information from a signal using a parametric method can be ill-posed or ill-conditioned. Also, sampling a signal below the Shannon-Nyquist rate causes aliasing, in other words different frequencies to become indistinguishable.

We present a parametric method to retrieve high-resolution information from coarse-scale measurements, while maintaining the use of a regular downsampling scheme. We exploit, rather than avoid, aliasing in order to regularize the problem statement and increase the resolution.

Our technique is very versatile because it can be combined with different existing multi-exponential analysis methods (matrix pencil, MUSIC, ESPRIT, ...). It seems to be especially useful in the presence of clusters of frequencies that are difficult to distinguish from one another.

Keywords

Exponential analysis, parametric method, Prony’s method, sub-Nyquist sampling, uniform sampling, sparse interpolation, signal processing.

1. Introduction

Estimating the fine scale spectral information of an exponential sum plays an important role in many signal processing applications. The problem of superresolution [2, 15] has therefore recently received considerable attention.

Despite its computational efficiency and wide applicability, the often used Fourier transform (FT) has some well-known limitations, such as its limited resolution and the leakage in the frequency domain. These restrictions complicate the analysis of signals falling exponentially with time. Fourier analysis, which represents a signal as a sum of periodic functions, is not very well suited for the decomposition of aperiodic signals, such as exponentially decaying ones. The damping causes a broadening of the spectral peaks, which in its turn leads to the peaks overlapping and masking the smaller amplitude peaks. The latter are important for the fine level signal classification.

Signals that fall exponentially with time appear, for instance, in transient detection, motor fault diagnosis, electrophysiology, magnetic resonance and infrared spectroscopy, vibration and seismic data analysis, music signal processing, corrosion rate and crack initiation modelling, electronic odour recognition, typed keystroke recognition, nuclear
science, liquid identification, direction of arrival estimation in sensor networks, and so on.

A different approach to spectral analysis is offered, among others, by parametric methods. However, parametric methods often suffer from ill-posedness and ill-conditioning, particularly in the case of clustered frequencies [11]. In general, parametric methods also require prior knowledge of the model order. Widely used parametric methods assuming a multi-exponential model include MUSIC [18], ESPRIT [17] and the matrix pencil algorithm [13].

In general, parametric methods as well as the FT, sample at a rate dictated by the Shannon-Nyquist theorem [16, 19]. It states that the sampling rate needs to be at least twice the maximum frequency present in the signal. A coarser time grid causes aliasing, identifying higher frequencies with lower frequencies without being able to distinguish between them. Conventional measurement systems, as used in modern consumer electronics, biomedical monitoring and medical imaging devices, are all based on this theorem.

In the past decade, alternative approaches have proved that signal reconstruction is also possible from sub-Nyquist measurements, if additional information on the structure of the signal is known, such as its sparsity. Many signals are indeed sparse in some domain such as time, frequency or space, meaning that most of the samples of either the signal or its transform in another domain can be regarded as zero. Among others, we refer to compressed sensing [3, 9], finite rate of innovation [22], the use of coprime arrays in DOA [21, 20].

The ultimate goal is to retrieve fine-scale information directly from coarse-scale measurements acquired at a slower information rate, in function of the sparsity and not the bandwidth of the signal. We offer a technique that allows to overcome the Shannon-Nyquist sampling rate limitation and at the same time improves the conditioning of the numerical linear algebra problems involved. The technique is exploiting aliasing rather than avoiding it and maintains a regular sampling scheme [5, 6]. It relies on the concept of what we call identification shift [6, 5], which is the additional sampling at locations shifted with respect to the original locations in order to overcome any ambiguity in the analysis.

The paper is organized as follows. Exponential analysis following Shannon-Nyquist sampling is repeated in Section 2 and generalized to sub-Nyquist sampling in Section 3. Since a sub-Nyquist rate can cause terms to collide at the time of the sampling, we explain how to unravel collisions in Section 4. Such collisions are very unlikely to happen in practice of course. In Section 5 numerical examples illustrate both the collision-free situation and the case in which the collision of terms happens. The numerical examples at the same time illustrate:

- how the method regularizes a problem statement that is numerically ill-conditioned because of the presence of frequency clusters,
- that it can be combined with an existing implementation of a multi-exponential spectral analysis (we used ESPRIT for the purpose).
To conclude, we mention that the current identification shift idea that allows to overcome any ambiguity from aliasing, can easily be combined with the similar use of identification shifts in the multivariate setting [8], where these are used to overcome any ambiguity introduced by taking projections.

2. The multi-exponential model

In order to proceed we introduce some notations. Let $\psi_i, \omega_i, \beta_i$ and $\gamma_i$ respectively denote the damping, frequency, amplitude and phase in each component of the signal

$$\phi(t) = \sum_{i=1}^{n} \alpha_i \exp(\phi_i t), \quad \alpha_i = \beta_i \exp(i \gamma_i), \quad \phi_i = \psi_i + i 2 \pi \omega_i.$$  

(1)

For the moment, we assume that the frequency content is limited by [16, 19]

$$|\Im(\phi_i)|/(2\pi) = |\omega_i| < \Omega/2, \quad i = 1, \ldots, n,$$

and we sample $\phi(t)$ at the equidistant points $t_j = j\Delta$ for $j = 0, 1, \ldots, 2n - 1, \ldots$ with $\Delta \leq 1/\Omega$. In the sequel we denote

$$f_j := \phi(t_j), \quad j = 0, 1, \ldots, 2n - 1, \ldots$$

The aim is to find the model order $n$, and the parameters $\phi_1, \ldots, \phi_n$ and $\alpha_1, \ldots, \alpha_n$ from the measurements $f_0, \ldots, f_{2n}, \ldots$. We further denote

$$\lambda_i := \exp(\phi_i \Delta), \quad i = 1, \ldots, n.$$  

With

$$H^{(k)}_n := \begin{pmatrix} f_k & \cdots & f_{k+n-1} \\ \vdots & \ddots & \vdots \\ f_{k+n-1} & \cdots & f_{k+2n-2} \end{pmatrix}, \quad k \geq 0, \quad n \geq 1,$$

the $\lambda_i$ are retrieved [13] as the generalized eigenvalues of the problem

$$H^{(1)}_n v_i = \lambda_i H^{(0)}_n v_i, \quad i = 1, \ldots, n,$$  

(2)

where $v_i$ are the generalized right eigenvectors. From the values $\lambda_i$, the complex numbers $\phi_i$ can be retrieved uniquely because of the restriction $|\Im(\phi_i \Delta)| < \pi$.  

In the absence of noise, the exact value for $n$ can be deduced from [12, p. 603] (for a detailed discussion see [14])

$$\det H^{(k)}_n \neq 0, \quad k \geq 0$$

$$\det H^{(k)}_\nu = 0, \quad \nu > n, \quad k \geq 0.$$  

(3)

In the presence of noise and/or clusters of eigenvalues, this technique is not very reliable though. But some convergence property can be used instead [4].
Finally, the $\alpha_i$ are computed from the interpolation conditions

$$\sum_{i=1}^{n} \alpha_i \exp(\phi_i t_j) = f_j, \quad j = 0, \ldots, 2n - 1,$$

either by solving the system in the least squares sense, in the presence of noise, or by solving a subset of $n$ (consecutive) interpolation conditions in case of a noise-free $\phi(t)$. Also, $n$ can everywhere be replaced by $N > n$, in order to model noise on the data by means of some additional $N - n$ noise terms in (1). Note that

$$\exp(\phi_i t_j) = \lambda_i^j$$

and that the coefficient matrix of (4) is therefore a Vandermonde matrix. It is well-known that the conditioning of structured matrices is something that needs to be monitored [1, 10].

Without loss of generality we assume in the sequel that $0 \leq \omega_i < \Omega \in \mathbb{N}, i = 1, \ldots, n$ instead of $|\omega_i| < \Omega/2, i = 1, \ldots, n$. Also we assume in Section 3 that $n$ is known or correctly detected as indicated in [4]. In Section 4 we explain how to detect $n$ concurrently with the computation of the $\phi_i$ and $\alpha_i$ from sub-Nyquist data.

3. Sub-Nyquist multi-exponential analysis

Some basic result is first deduced for $n = 1$. Afterwards this result is made use of for general $n$. The latter however, demands additional developments.

3.1 Dealing with a single frequency ($n = 1$)

At first we deal with some simple mathematical results, without caring about computational issues. When

$$\phi(t) = \alpha \exp(\psi t + i2\pi \omega t), \quad 0 \leq \omega < \Omega,$$

and $\phi(t)$ is sampled at $t_j = 0, \Delta, 2\Delta, \ldots$, with for simplicity $\Delta = 1/\Omega$, then $\omega$ can uniquely be determined in $[0, \Omega)$ from the samples. No periodicity problem occurs since $\omega \Delta < 1$ in the generalized eigenvalue

$$\lambda = \exp(\phi \Delta) = \exp(\psi \Delta) \exp(i2\pi \omega \Delta).$$

When $\phi(t)$ is sampled at multiples $t_{r_1j} = 0, r_1 \Delta, 2r_1 \Delta, \ldots$ with $1 < r_1 \in \mathbb{N}$, then there exist $r_1$ solutions for $\omega$ in $[0, \Omega)$ since $0 \leq 2\pi \omega r_1 \Delta < 2r_1 \pi$. If $\phi(t)$ is also sampled at $t_{r_2j} = 0, r_2 \Delta, 2r_2 \Delta, \ldots$ with $0 < r_2 \in \mathbb{N}$, then one obtains another set containing $r_2$ solutions for $\omega$. Each solution set is extracted from the respective generalized eigenvalues $\exp(\psi r_m \Delta) \exp(i2\pi \omega r_m \Delta), m = 1, 2$ satisfying (2) where the first generalized eigenvalue problem is set up with the samples $f_{r_1j} = \phi(0), \phi(r_1 \Delta), \phi(2r_1 \Delta), \ldots$ and the second generalized eigenvalue problem with the samples $f_{r_2j} = \phi(0), \phi(r_2 \Delta), \phi(2r_2 \Delta), \ldots$. In our write-up we have chosen not to add an index $r$ to the notation of the Hankel matrices $H_n^{(0)}$ and $H_n^{(1)}$ when they are filled with samples taken at multiples $t_{rj} = 0, r \Delta, 2r \Delta, \ldots$ in order to not overload the notation. From the context it is always clear which sequence of samples is meant.

It is easy to show that, if in addition $\gcd(r_1, r_2) = 1$, then $\omega$ is the unique intersection of the two solution sets.
Lemma 1. Let $0 \leq \omega < \Omega$ and $\Omega, r_1, r_2$ be nonzero positive integers. If $\gcd(r_1, r_2) = 1$ and $\Delta = 1/\Omega$, then from the values $\exp(i2\pi\omega r_1 \Delta)$ and $\exp(i2\pi\omega r_2 \Delta)$ the frequency $\omega$ can uniquely be recovered in $[0, \Omega)$.

Proof. From the generalized eigenvalue $\lambda^{r_1} = \exp(\psi r_1 \Delta) \exp(i2\pi r_1 \Delta)$ we extract $r_1$ solutions for $\omega$:

$$\omega = \omega^{(1)} + k \frac{\Omega}{r_1}, \quad 0 \leq \omega^{(1)} < \frac{\Omega}{r_1}, \quad k = 0, \ldots, r_1 - 1. \quad (5)$$

From the value $\lambda^{r_2} = \exp(\psi r_2 \Delta) \exp(i2\pi r_2 \Delta)$ we extract $r_2$ solutions for $\omega$:

$$\omega = \omega^{(2)} + \ell \frac{\Omega}{r_2}, \quad 0 \leq \omega^{(2)} < \frac{\Omega}{r_2}, \quad \ell = 0, \ldots, r_2 - 1. \quad (6)$$

Note that the frequency $\omega$ we are trying to identify satisfies both (5) and (6). Remains to show that the common solution to (5) and (6) is unique. Suppose we have two distinct values for $\omega$ that both satisfy (5) and (6). This implies that there exist two distinct $0 \leq k_1, k_2 < r_1$ such that

$$\omega^{(1)} + k_1 \frac{\Omega}{r_1} = \omega^{(2)} + \ell_1 \frac{\Omega}{r_2},$$

$$\omega^{(1)} + k_2 \frac{\Omega}{r_1} = \omega^{(2)} + \ell_2 \frac{\Omega}{r_2}, \quad (7)$$

with $0 \leq \ell_1, \ell_2 < r_2$ and $\ell_1 \neq \ell_2$ because $k_1 \neq k_2$. From (7) we deduce

$$k_1 - k_2 = \frac{(\ell_1 - \ell_2)r_1}{r_2} \neq 0.$$ 

Hence $r_2$ divides $\ell_1 - \ell_2$ because $\gcd(r_1, r_2) = 1$. Since $\ell_1 - \ell_2$ is bounded in absolute value by $r_2 - 1$, this is a contradiction. \qed

Furthermore, the element $\omega$ in the intersection can be obtained from the Euclidean algorithm.

Lemma 2. Let $0 \leq \omega < \Omega$ and $\Omega, r_1, r_2$ be nonzero positive integers. If $\gcd(r_1, r_2) = 1$ and $\Delta = 1/\Omega$, then from the values $\exp(i2\pi\omega r_1 \Delta)$ and $\exp(i2\pi\omega r_2 \Delta)$ the frequency $\omega \in [0, \Omega)$ is obtained as

$$\left( p_1 \frac{\Log(\exp(i2\pi\omega r_1 \Delta))}{12\pi} + p_2 \frac{\Log(\exp(i2\pi\omega r_2 \Delta))}{12\pi} \right) \Omega = \omega + h\Omega, \quad h \in \mathbb{Z}, \quad (8)$$

where $p_1 r_1 + p_2 r_2 = 1 \mod \Omega$ with $p_1, p_2 \in \mathbb{Z}$ and $\Log(\cdot)$ denotes the principal branch of the complex logarithm.

Proof. We use the same notation as in the proof of Lemma 1. So we have

$$\omega = \omega^{(1)} + k \frac{\Omega}{r_1}, \quad 0 \leq \omega^{(1)} < \frac{\Omega}{r_1},$$

$$\omega = \omega^{(2)} + \ell \frac{\Omega}{r_2}, \quad 0 \leq \omega^{(2)} < \frac{\Omega}{r_2}.$$
Then
\[
\Omega \frac{\log \left( \exp \left( i 2 \pi \omega_1 r_1 \Delta \right) \right)}{12\pi} = \omega^{(1)}, \quad \Omega \frac{\log \left( \exp \left( i 2 \pi \omega_2 r_2 \Delta \right) \right)}{12\pi} = \omega^{(2)},
\]
and
\[
\left( p_1 \frac{\log \left( \exp \left( i 2 \pi \omega_1 r_1 \Delta \right) \right)}{12\pi} + p_2 \frac{\log \left( \exp \left( i 2 \pi \omega_2 r_2 \Delta \right) \right)}{12\pi} \right) \Omega = (p_1 r_1 + p_2 r_2) \omega - (p_1 k + p_2 \ell) \Omega
\]
\[
= \omega - (p_1 k + p_2 \ell) \Omega,
\]
in which \( p_1 k + p_2 \ell \) is an integer. \( \square \)

When the integers \( p_1 \) and \( p_2 \) are small this method is very useful. Otherwise one has to be careful about the numerical stability of (8). One can of course experiment with different \( r_1 \) and \( r_2 \) values to ensure small \( p_1 \) and \( p_2 \) values.

### 3.2 Dealing with several terms (\( n > 1 \))

When \( \phi(t) \) contains several terms, then we obtain \( n \) solution sets for the \( \omega_i, i = 1, \ldots, n \) from the first batch of evaluations at multiples of \( r_1 \Delta \) and another \( n \) solution sets for these frequencies from the second batch of samples at multiples of \( r_2 \Delta \). But now we are facing the problem of correctly matching the solution set from the first batch to the solution set from the second batch that refer to the same \( \omega_i \). Of course, we want to avoid such combinatorial steps in our algorithm. To solve this problem we are going to choose the second batch of sampling points in a smarter way.

Before we proceed we assume that we don’t have \( \exp(\phi_k r \Delta) = \exp(\phi_\ell r \Delta) \) for distinct \( k \) and \( \ell \) with \( 1 \leq k, \ell \leq n \). In Section 4 we explain how to deal with the collision of terms, which we exclude in the sequel of this section.

The sampling strategy that we propose is the following. Sampling at \( t_{rj} = 0, r \Delta, 2r \Delta, \ldots \) with fixed \( 1 < r \in \mathbb{N} \), gives us only aliased values for \( \omega_i \), obtained from \( \log(\exp(i 2 \pi \omega r \Delta)) \). This aliasing can be fixed at the expense of the following additional samples. In what follows \( n \) can also everywhere be replaced by \( N > n \) when using \( N - n \) additional terms in (1) to model the noise.

To fix the aliasing, we add \( n \) samples to the already collected \( f_0, f_r, \ldots, f_{(2n-1)r} \), namely at the shifted points
\[
t_{rj + \rho} = j r \Delta + \rho \Delta, \quad r, \rho \text{ fixed},
\]
\[
j = h, \ldots, h + n - 1, \quad 0 \leq h \leq n.
\]

An easy choice for \( \rho \) is a number mutually prime with \( r \). For the most general choice allowed, we refer to [7]. Also, the shifted points need not be consecutive, but for ease of notation we assume this for now.

From the samples \( f_0, f_r, \ldots, f_{(2n-1)r} \) we first compute the generalized eigenvalues \( \lambda^r_i = \exp(\phi_i r \Delta) \) and the coefficients \( \alpha_i \) going with \( \lambda^r_i \) in the model
\[
\phi(j r \Delta) = \sum_{i=1}^{n} \alpha_i \exp(\phi_i j r \Delta)
\]
\[
= \sum_{i=1}^{n} \alpha_i \lambda^r_i, \quad j = 0, \ldots, 2n - 1.
\]

(9)
So we know which coefficient $\alpha_i$ goes with which generalized eigenvalue $\lambda_i^r$, but we just cannot identify the correct $\Im(\phi_i)$ from $\lambda_i^r$. The samples $f_{jr+\rho}$ at the additional points $t_{rj+\rho}$ satisfy

$$
\phi(jr\Delta + \rho\Delta) = \sum_{i=1}^{n} \alpha_i \exp(\phi_i(jr + \rho)\Delta)
$$

$$
= \sum_{i=1}^{n} (\alpha_i \lambda_i^0) \lambda_i^jr,
$$

(10)

$$
\quad j = h, \ldots, h + n - 1, \quad 0 \leq h \leq n,
$$

which can be interpreted as a linear system with the same coefficient matrix entries as (9), but now with a new left hand side and unknowns $\alpha_1 \lambda_1^0, \ldots, \alpha_n \lambda_n^0$ instead of $\alpha_1, \ldots, \alpha_n$. And again we can associate each computed coefficient $\alpha_i \lambda_i^0$ with the proper generalized eigenvalue $\lambda_i^r$. Then by dividing the $\alpha_i \lambda_i^0$ computed from (10) by the $\alpha_i$ computed from (9), for $i = 1, \ldots, n$, we obtain from $\lambda_i^0$ a second set of $\rho$ plausible values for $\omega_i$. Because of the fact that we choose $\rho$ and $r$ relatively prime, the two sets of plausible values for $\omega_i$ have only one value in their intersection, as explicated in Lemma 1 and 2. Thus the aliasing problem is solved.

4. When aliasing causes terms to collide

When $\exp(\phi_k r\Delta) = \exp(\phi_\ell r\Delta)$ with $k \neq \ell$ then different exponential terms in (1) collide into one term as a consequence of the undersampling and the aliasing effect. Note that then the moduli $\exp(\psi_k r\Delta) = \exp(\psi_\ell r\Delta)$ and consequently $\psi_k = \psi_\ell$. As long as $\psi_k \neq \psi_\ell$ exponential terms can be distinguished on the basis of their modulus. So our focus is on the situation where

$$
\phi_k = \psi_k + i2\pi\omega_k \neq \phi_\ell = \psi_\ell + i2\pi\omega_\ell, \quad \psi_k = \psi_\ell, \quad r\omega_k = r\omega_\ell + h\Omega, \quad h \in \mathbb{Z}.
$$

Since terms can collide when subsampling, their correct number $n$ may not be revealed when sampling at multiples of $r\Delta$, in other words, when sampling at the rate $\Omega/r$ instead of $\Omega$. Let us assume that (3), or its practical implementation in [4] on $N \times N$ Hankel matrices with $N > n$, reveals a total of $n_0$ terms after the first batch of evaluations at $t_{rj} = 0, r\Delta, 2r\Delta, \ldots$ with fixed $r$. We call $\lambda_i^{(0)}$ the $n_0$ generalized eigenvalues of (2) computed from the $f_{jr}$ as in Section 3. Since some of the terms in (1) may have collided, we have

$$
\phi(t_{rj}) = \sum_{i=1}^{n_0} \alpha_i^{(0)} \exp(\phi_i^{(0)} t_{rj})
$$

with

$$
\lambda_i^{(0)} = \exp(\phi_i^{(0)} r\Delta), \quad i = 1, \ldots, n_0,
$$

and some of the $\alpha_i^{(0)}$ being sums of the $\alpha_i$ from (1). We remark that $n_0 \leq n$ and that the $\phi_i^{(0)}$ are definitely among the $n$ parameters $\phi_i$ in (1). Without loss of generality we
assume that the colliding terms are successive,

\[
\begin{pmatrix}
\alpha_{1}^{(0)} \\
\vdots \\
\alpha_{n_0}^{(0)}
\end{pmatrix} = \begin{pmatrix}
\alpha_{h_1} + \cdots + \alpha_{h_2-1} \\
\vdots \\
\alpha_{h_{n_0}} + \cdots + \alpha_{h_{n_0+1}-1}
\end{pmatrix},
\]

\[h_1 = 1, \quad h_j \leq h_{j+1}, \quad 1 \leq j \leq n_0, \quad h_{n_0+1} = n + 1.
\]

In brief, when collisions occur, the computations return the results

\[
\alpha_i^{(0)} = \sum_{\ell = h_i}^{h_i+1-1} \alpha_{\ell}, \quad i = 1, \ldots, n_0
\]

\[
\lambda_i^{(0)} = \lambda_{h_i}^{r} = \cdots = \lambda_{h_{i+1}-1}^{r}, \quad i = 1, \ldots, n_0.
\]

Note that only the nonzero \(\alpha_i^{(0)}\) are revealed. At the end of this section we explain how to deal with the additional problem where some of the \(\alpha_i\) cancel each other.

For the sake of completeness we explicit the linear system that delivered the \(\alpha_i^{(0)}\), namely

\[
\begin{pmatrix}
1 \\
\lambda_1^{(0)} \\
\vdots \\
\lambda_{n_0}^{(0)} \\
(\lambda_1^{(0)})_{n_0-1} \\
\vdots \\
(\lambda_{n_0}^{(0)})_{n_0-1}
\end{pmatrix}
\begin{pmatrix}
\alpha_1^{(0)} \\
\vdots \\
\alpha_{n_0}^{(0)}
\end{pmatrix} =
\begin{pmatrix}
f_0 \\
f_r + k_\rho \\
\vdots \\
f_{(n_0-1) r + k_\rho}
\end{pmatrix}
\]

or, as is most often the case, an overdetermined version of it. We now explain how to disentangle the collisions, again making use of some additional samples at shifted locations. Let \(r\) and \(\rho\) be fixed as before with \(\gcd(r, \rho) = 1\). If \(\gcd(r, \rho) > 1\) for some reason or because of a practical constraint, then the procedure may be an iterative one, as we indicate further below.

Let us sample \(\phi(t)\) at the shifted locations \(t_{rj + k_\rho} = (jr + k_\rho)\Delta, j = 0, \ldots, n_0 - 1, k \geq 1\). These sample values equal

\[
f_{j r + k_\rho} := \sum_{i=1}^{n_0} \left( \sum_{\ell = h_i}^{h_i+1-1} \alpha_{\ell} \exp(\phi_{\ell} k_\rho \Delta) \right) \exp(\phi_i^{(0)} j r \Delta).
\]

In (12) we abbreviate

\[
\alpha_i^{(1)}(k) := \sum_{\ell = h_i}^{h_i+1-1} \alpha_{\ell} \exp(\phi_{\ell} k_\rho \Delta), \quad i = 1, \ldots, n_0.
\]

For \(k = 0\) we have \(\alpha_i^{(1)}(0) = \alpha_i^{(0)}, i = 1, \ldots, n_0\). For fixed \(k > 0\) the values \(\alpha_i^{(1)}(k), i = 1, \ldots, n_0\) are obtained from (12) and

\[
\begin{pmatrix}
1 \\
\lambda_1^{(0)} \\
\vdots \\
(\lambda_1^{(0)})_{n_0-1} \\
\vdots \\
\vdots \\
(\lambda_{n_0}^{(0)})_{n_0-1} \\
\lambda_{n_0}^{(0)}
\end{pmatrix}
\begin{pmatrix}
\alpha_1^{(1)}(k) \\
\vdots \\
\alpha_{n_0}^{(1)}(k)
\end{pmatrix} =
\begin{pmatrix}
f_{k_\rho} \\
f_{r + k_\rho} \\
\vdots \\
f_{(n_0-1) r + k_\rho}
\end{pmatrix}
\]

(14)
or its least squares version. The Vandermonde coefficient matrix of (14) is the same as the one used to compute \( \alpha_i^{(0)}, i = 1, \ldots, n_0 \) in (11) from the samples \( f_{jr} \), which is the case \( k = 0 \). So the Vandermonde matrix is reused as it is independent of the index \( k \) in the right hand side and in the vector of unknowns.

When collecting in this way, for each \( 1 \leq i \leq n_0 \), the values \( \alpha_i^{(1)}(0), \alpha_i^{(1)}(1), \alpha_i^{(1)}(2), \ldots \) we have a separate exponential analysis problem per \( i \), namely to identify the number of terms in \( \alpha_i^{(1)}(k) \) in (13). Note that the sampling rate used to collect the \( \alpha_i^{(1)}(k) \) is \( \Omega/\rho \). Now we fix \( 1 \leq i \leq n_0 \) and proceed.

When the values \( \alpha_i^{(1)}(k) \) take the place of the values \( f_k \) in (2) and \( h_{i+1} - h_i \) that of \( n \), then the generalized eigenvalue problem constructed with them delivers the components \( \lambda_{\ell}^{(1)} = \exp(\phi_{\ell} \rho \Delta) \) in (13) and the respective Vandermonde system (13) delivers the \( \alpha_{\ell}, \ell = h_i, \ldots, h_{i+1} - 1 \). Both can again be set up in a least squares sense, in a similar way as for the determination of the \( \lambda_i^{(0)} \) and \( \alpha_i^{(0)} \). As shown in Lemma 1, the \( \alpha_i^{(1)}(k) \) are fully disentangled and all \( n \) terms in (1) are identified when \( \gcd(r, \rho) = 1 \), which is what we try to achieve in practice. Then for each \( i = 1, \ldots, n_0 \),

\[
\lambda_{\ell}^{(1)} = \exp(\phi_{\ell} \rho \Delta) = \lambda_{\ell}^{\rho}, \quad \ell = h_i, \ldots, h_{i+1} - 1.
\]

Together with

\[
\lambda_i^{(0)} = \exp(\phi_{hi} r \Delta) = \lambda_i^{r}, \quad \ell = h_i, \ldots, h_{i+1} - 1,
\]

we have what we need to identify the \( \phi_i, i = 1, \ldots, n \) using Lemma 2. An illustration of the above is presented in Section 5.2. However, when for one or other reason \( \gcd(r, \rho) = s \neq 1 \) then the above procedure needs to be repeated with \( r \) replaced by \( s \) and \( \rho \) replaced by a suitable \( \sigma \). Then again additional samples are collected at shifted locations \( t_{sj+\sigma k} = (js + k\sigma)\Delta \), namely

\[
f_{js+k\sigma} := \phi((js + k\sigma)\Delta),
\]

and the procedure is repeated from (12) on. The procedure ends when \( \gcd(r, \rho, \sigma) = 1 \), otherwise it continues as described.

To conclude this section, we discuss the special situation where some of the terms \( \alpha_i \exp(\phi_{i}jr \Delta) \) cancel each other when evaluating (1) at the \( tr_j \), a situation which is illustrated in Section 5.3. In order to observe exact cancellations the example will treat a purely noise free multi-exponential model.

So at the first batch of evaluations \( f_{jr} \), in addition to collision, one encounters cancellation for one or more indices \( i, 1 \leq i \leq n_0 \). The fundamental question is whether the \( \alpha_i^{(1)}(k) \) can continue to evaluate to zero for all \( k \) in the second shifted batch of evaluations \( f_{jr+k\rho} \) when \( \gcd(r, \rho) = 1 \). On the side, we remark that the cancellation can only persist if the \( \phi_{\ell} \) in (13) have the same decay rate. For the index \( i \) which falls victim to
cancellation, we consider the following square Vandermonde system which is obtained from (13) for fixed \( i \) and increasing \( k \),

\[
\begin{pmatrix}
1 & \ldots & 1 \\
\exp(\phi_i \rho \Delta) & \ldots & \exp(\phi_{i+1} \rho \Delta) \\
\vdots & & \vdots \\
\exp(\phi_i (h_{i+1} - h_i) \rho \Delta) & \ldots & \exp(\phi_{i+1-1} (h_{i+1} - h_i) \rho \Delta)
\end{pmatrix}
\begin{pmatrix}
\alpha_{h_i} \\
\vdots \\
\alpha_{h_{i+1}-1}
\end{pmatrix}
= \begin{pmatrix}
\alpha_i^{(1)}(0) \\
\vdots \\
\alpha_i^{(1)}(h_{i+1} - h_i)
\end{pmatrix}.
\]

If the right hand side of this small linear system consists of all zeroes, then we must conclude incorrectly that

\[
\alpha_{h_i} = \cdots = \alpha_{h_{i+1}-1} = 0
\]

because the \((h_{i+1} - h_i) \times (h_{i+1} - h_i)\) Vandermonde coefficient matrix is regular, as we show in Lemma 3 below. So from this we know that the evaluation of \( \alpha_i^{(1)}(k) \) to zero cannot persist up to and including \( k = h_{i+1} - h_i \).

**Lemma 3.** Let for \( \phi_k \neq \phi_\ell \) and \( r \neq 0 \) hold that \( \exp(\phi_k r \Delta) = \exp(\phi_\ell r \Delta) \). If \( \gcd(r, \rho) = 1 \) then

\[
\exp(\phi_k \rho \Delta) \neq \exp(\phi_\ell \rho \Delta).
\]

**Proof.** As pointed out it is sufficient to deal with the imaginary parts of \( \phi_k \) and \( \phi_\ell \). We use a similar notation as in Lemma 1. The proof is by contraposition. From \( \exp(\phi_k r \Delta) = \exp(\phi_\ell r \Delta) \) and \( \exp(\phi_k \rho \Delta) = \exp(\phi_\ell \rho \Delta) \) we find that there exist integers \( p_k, p_\ell, q_k, q_\ell \) such that

\[
\omega_k = \omega^{(1)} + \frac{p_k}{r}, \quad 0 \leq p_k \leq r - 1
\]

\[
\omega_\ell = \omega^{(1)} + \frac{p_\ell}{r}, \quad 0 \leq p_\ell \leq r - 1
\]

\[
\omega_k = \omega^{(2)} + \frac{q_k}{\rho}, \quad 0 \leq q_k \leq \rho - 1
\]

\[
\omega_\ell = \omega^{(2)} + \frac{q_\ell}{\rho}, \quad 0 \leq p_k \leq \rho - 1.
\]

Then

\[
\omega_k - \omega_\ell = \frac{(p_k - p_\ell) \Omega}{r} = \frac{(q_k - q_\ell) \Omega}{\rho}
\]

or

\[
p_k - p_\ell = \frac{q_k - q_\ell}{\rho} r,
\]

which is a contradiction since the left hand side is an integer and \( q_k - q_\ell \) in the right hand side is in absolute value bounded by \( \rho - 1 \). □
5. Numerical illustration

We illustrate the working of (9) and (10) from Section 3 and that of (11) and (14) from Section 4 on two examples. In the former numerical example the undersampling will not cause collisions, while in the latter illustration it will. In addition, we show the detection of terms that have not only collided but entirely vanished in the first sampling at multiples of $r\Delta$.

5.1 Collision-free example

For our first example the $\alpha_i$ and $\phi_i$ are given in Table 1. We take $\Omega = 1000$ and $\Delta = 1/\Omega$. The $n = 20$ frequencies $\omega_i$ form 5 clusters, as is apparent from the FT, computed from 1000 samples and shown in Figure 1. For completeness we graph the signal in Figure 2. In Figure 3 we show the generalized eigenvalues $\lambda_i = \exp(\phi_i \Delta)$, $i = 1, \ldots, 20$ computed from the noise-free samples, to illustrate the ill-conditioning of the problem as a result of the clustering of the frequencies.

![Figure 1: Real (blue) and imaginary (red) part of the FT.](image1.png)

![Figure 2: Real (blue) and imaginary (red) part of the signal.](image2.png)
To (1) we add white Gaussian noise with SNR = 32. For comparison with our new method, we show in Figure 4 the \((\omega_i, \beta_i)\) results computed by means of ESPRIT using 270 samples, namely \(f_0, \ldots, f_{269}\) for a signal space of dimension 20 and a noise space of dimension 31 (so total dimension \(N = 51\) for which the best ESPRIT results were obtained). The true \((\omega_i, \beta_i)\) couples from Table 1 are indicated using black circles. The ESPRIT output is indicated using red bullets. So ideally every black circle should be hit by a red bullet. The ill-conditioning has clearly created a serious problem in identifying the individual input frequencies and amplitudes.

Next we choose \(r = 11\) and \(\rho = 5\). The originally clustered eigenvalues are now much better separated. To illustrate this we show in Figure 5 the noise-free generalized eigenvalues \(\lambda^r_i, i = 1, \ldots, 20\) of the \(r\)-fold undersampled exponential analysis problem.

With the noisy samples, we now take \(N = 90 > n = 20\) and set up the \(90 \times 90\) generalized eigenvalue problem (2) with the samples \(f_{jr}, j = 0, \ldots, 179\) and the \(180 \times 90\) Vandermonde system (9) that respectively deliver the \(\lambda^r_i\) and the \(\alpha_i\) for \(i = 1, \ldots, N\). With the samples \(f_{jr+\rho}, j = 0, \ldots, 89\) we set up the \(90 \times 90\) linear system (10) from which we compute the \(\alpha_i \lambda^\rho_i, i = 1, \ldots, 90\) and subsequently the \(\lambda^\rho_i\). This brings our
total number of samples used also to 270, comparable to the ESPRIT procedure. An advantage for ESPRIT is that the signal has less decayed in the first 270 samples, compared to the 270 samples used here. Using the Euclidean algorithm, as explicited in Lemma 2, we recover from $\lambda^r_i$ and $\lambda^\rho_i$ the true frequencies $\omega_i$ with $p_1 = 1, p_2 = -2$ and $p_1 r + p_2 \rho = 1$. With the new method we find the $\omega_i, \beta_i$ couples shown as blue dots in Figure 6. In Figure 6 we have also provided enlarged insets so that the reader can clearly count the number of frequencies retrieved in each cluster, which is the correct number when comparing to the input values in Table 1. Clearly Figure 6 is a tremendous improvement over Figure 4.

Figure 5: Generalized eigenvalues $\lambda^r_i$ for (1) with data from Table 1 and $r = 11$.

Figure 6: Output $(\omega_i, \beta_i), i = 1, \ldots, 20$ computed from 270 samples $f_{jr+\rho}, r = 11, \rho = 5$.

5.2 Example where collisions occur

In Table 2 we list the $\alpha_i$ and $\phi_i$ of an exponential model, chosen in such a way that the aliasing causes terms to collide. This enables us to illustrate the workings of the technique explained in Section 4. Moreover, some of the terms in the $\alpha_i^{(1)}$ will cancel and so there is an additional challenge to retrieve the correct number of terms in $\alpha_i^{(0)}$. The bandwidth is again $\Omega = 1000$ and we take $\Delta = 1/\Omega$ and $r = 100$. We add
white Gaussian noise to the samples with SNR = 20 and start our computations. When subsampling, the 6 terms collide into 3, as indicated in Figure 7 by the singular value decomposition of $H_N^{(0)}$ with $N = 30$, which reveals its numerical rank. Actually

$$\phi(t_{rj}) = (\alpha_1 + \alpha_2 + \alpha_3) \exp(\phi_1 j r \Delta) + \alpha_4 \exp(\phi_4 j r \Delta) + (\alpha_5 + \alpha_6) \exp(\phi_5 j r \Delta).$$

We recall that $H_N^{(0)}$ is filled with the samples $f_{jr}, j = 0, \ldots, 59$ and not with the samples $f_j, j = 0, \ldots, 59$.

![Figure 7: SVD of $H_N^{(0)}$ for (1) with data from Table 2 and $r = 100$.](image)

We set up the $30 \times 30$ generalized eigenvalue problem (2) with the samples $f_{rj}, j = 0, \ldots, 59$ and the $60 \times 30$ Vandermonde system (9) that respectively deliver the $\lambda_i$ and the $\alpha_i$ for $i = 1, \ldots, N$. When retaining the components with largest $|\alpha_i|$, we find

$$\lambda_1^{(0)} \approx 0.36845 + 0.93042i$$
$$\lambda_2^{(0)} \approx 0.36745 - 0.92977i$$
$$\lambda_3^{(0)} \approx -0.72761 - 0.68801i$$

and

$$\alpha_1^{(0)} = \alpha_1 + \alpha_2 + \alpha_3 \approx 17.718 + 0.25273i$$
$$\alpha_2^{(0)} = \alpha_4 \approx 16.126 + 0.057118i$$
$$\alpha_3^{(0)} = \alpha_5 + \alpha_6 \approx 4.5732 - 0.53331i$$

At this point we have not yet been able to recover the correct $\lambda_i$ and $\alpha_i$ for the signal defined by the parameters in Table 2 (we have unearthed only 3 terms instead of 6) because of two reasons. First, the subsampling creates an aliasing effect and second the aliasing causes frequencies to collide. As explained in Section 4, we can disentangle the information in the collisions from more values $\alpha_i^{(1)}(k), k = 1, 2, \ldots,$ where $\alpha_i^{(0)} = \alpha_i^{(1)}(0)$, simply because the $\alpha_i^{(1)}(k)$ are themselves linear combinations of exponentials. To this end we now choose $\rho = 133$ and we set up the Vandermonde systems (14),

$$\begin{pmatrix}
\lambda_1^{(0)} & \cdots & \lambda_1^{(0)} \\
\lambda_2^{(0)} & \cdots & \lambda_2^{(0)} \\
\vdots & \ddots & \vdots \\
(\lambda_1^{(0)})^9 & \cdots & (\lambda_3^{(0)})^9 \\
\end{pmatrix} \begin{pmatrix}
\alpha_1^{(1)}(k) \\
\alpha_2^{(1)}(k) \\
\alpha_3^{(1)}(k) \\
\end{pmatrix} = \begin{pmatrix}
f_{k\rho} \\
f_{r+k\rho} \\
\vdots \\
f_{9r+k\rho} \\
\end{pmatrix}, \quad k = 1, \ldots, 11.$$
In total so far 170 samples are used. A singular value analysis of the Hankel matrices
\[
\begin{pmatrix}
\alpha_i^{(1)}(0) & \alpha_i^{(1)}(1) & \ldots & \alpha_i^{(1)}(5) \\
\alpha_i^{(1)}(1) & \alpha_i^{(1)}(2) & \ldots & \alpha_i^{(1)}(6) \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_i^{(1)}(5) & \alpha_i^{(1)}(6) & \ldots & \alpha_i^{(1)}(10)
\end{pmatrix}, \quad i = 1, 2, 3
\]
reveal the number of components that one can distinguish in and consequently extract from the \(\alpha_i^{(1)}(k)\). The numbers are respectively 3, 2, 1 for \(i = 1, 2, 3\) and so \(h_1 = 1, h_2 = 4, h_3 = 6, h_4 = 7\). For \(i = 1, 2, 3\) the generalized eigenvalue problems
\[
\begin{pmatrix}
\alpha_i^{(1)}(1) & \alpha_i^{(1)}(2) & \ldots & \alpha_i^{(1)}(6) \\
\alpha_i^{(1)}(2) & \alpha_i^{(1)}(3) & \ldots & \alpha_i^{(1)}(7) \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_i^{(1)}(6) & \alpha_i^{(1)}(7) & \ldots & \alpha_i^{(1)}(11)
\end{pmatrix} v_\ell = \lambda^{(1)} v_\ell
\]
\[
\begin{pmatrix}
\alpha_i^{(1)}(0) & \alpha_i^{(1)}(1) & \ldots & \alpha_i^{(1)}(5) \\
\alpha_i^{(1)}(1) & \alpha_i^{(1)}(2) & \ldots & \alpha_i^{(1)}(6) \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_i^{(1)}(5) & \alpha_i^{(1)}(6) & \ldots & \alpha_i^{(1)}(10)
\end{pmatrix} v_\ell
\]
reveal the \(\lambda^{(1)}_\ell = \exp(\phi_\ell \rho \Delta), \ell = h_i, \ldots, h_{i+1} - 1\). The respective Vandermonde systems with unknowns \(\alpha_{h_i}, \ldots, \alpha_{h_{i+1} - 1}\) and right hand sides \(\alpha_i^{(1)}(0), \ldots, \alpha_i^{(1)}(11)\) reveal the \(\alpha_\ell, \ell = h_i, \ldots, h_{i+1} - 1\) in (13). Again we retain only the \(h_{i+1} - h_i\) components with largest \(|\alpha_\ell|\). From the \(\lambda^{(0)}_\ell = \exp(\phi_\ell r \Delta), \ell = h_i, \ldots, h_{i+1} - 1, i = 1, \ldots, n_0\) and \(\lambda^{(1)}_\ell = \exp(\phi_\ell r \Delta), \ell = 1, \ldots, n\) the imaginary part of \(\phi_i\) can be recovered as indicated in Lemma 2: with \(p_1 = 4\) and \(p_2 = -3\) we have \(p_1 r + p_2 \rho = 1\) and so
\[
\Im(\phi_\ell) := 4 \Arg(\lambda^{(0)}_\ell) \Omega - 3 \Arg(\lambda^{(1)}_\ell) \Omega + 2 \pi h \Omega, \quad 1 \leq \ell \leq 6, \quad h \in \mathbb{Z},
\]
where \(h\) is taken such that \(0 \leq \Im(\phi_\ell) < 2 \pi \Omega\). Eventually we unearth the following 6 \(\phi_i\) and \(\alpha_i\):
\[
\phi_1 \approx -0.021600 + i 2 \pi 192.29 \\
\phi_2 \approx -0.0085122 + i 2 \pi 289.87 \\
\phi_3 \approx -0.025728 + i 2 \pi 386.69 \\
\phi_4 \approx -0.066292 + i 2 \pi 538.18 \\
\phi_5 \approx -0.043745 + i 2 \pi 585.70 \\
\phi_6 \approx 0.0026126 + i 2 \pi 956.23
\]
and
\[
\alpha_1 \approx 19.011 + i 0.53818 \\
\alpha_2 \approx -20.481 + i 0.89352 \\
\alpha_3 \approx 21.445 - i 1.5790 \\
\alpha_4 \approx 5.8439 - i 0.035907 \\
\alpha_5 \approx 5.0770 + i 0.03562 \\
\alpha_6 \approx 10.758 - i 0.40878
\]
5.3 Example where cancellation occurs

The cancellation strategy is most clearly illustrated on a noise-free example, where exact cancellation happens. The actual occurrence of this situation in case of real-life data is extremely small, but we primarily want to show that the proposed sub-Nyquist method is capable of recovering from it.

Let

\[
\phi(t) = \exp(2\pi i 19.3t) - \exp(2\pi i 39.3t) + 2 \exp(2\pi i 71t) + \exp(2\pi i 95.2t),
0 \leq \omega_i < \Omega, \quad i = 1, \ldots, 4. \quad (15)
\]

We take \(\Omega = 100, \Delta = 0.01\) and sample \(f_j = \phi(j\Delta)\) for particular values of \(j\). With \(r = 5\) the first two terms, where \(\omega_1 = 19.3\) and \(\omega_2 = 39.3\), collide and cancel. So from the samples \(f_0, f_5, f_{10}, \ldots\) only two terms can be retrieved:

\[
\text{rank } H_3^{(0)} = \text{rank} \begin{pmatrix} f_0 & f_5 & f_{10} \\ \vdots & \vdots & \vdots \\ f_{10} & f_{15} & f_{20} \end{pmatrix} = 2.
\]

With \(\rho = 3\) we find that

\[
\text{rank } H_4^{(3)} = \text{rank} \begin{pmatrix} f_3 & f_8 & f_{13} & f_{18} \\ \vdots & \vdots & \vdots \\ f_{18} & \ldots & f_{33} \end{pmatrix} = 3.
\]

From Lemma 3 we know that we must have missed (at least) one term. Unfortunately we still have a collision, but no cancellation this time. From the shifted samples we obtain the \(\alpha_{h_i}^{(1)}(1)\), where we know from (15) that \(h_1 = 1, h_2 = 3, h_3 = 4\). Remember that in real-life these indices \(h_i\) are not known and need not be known. They are revealed as the algorithm progresses.

As described in Section 4 on the disentangling of collisions, we need to continue sampling at multiples of the shift, namely collect the \(f_{jr+k\rho}\) for \(k > 1\). In this way we obtain the values for \(\alpha_{i}^{(1)}(k), i = 1, 3, 4\) where \(\alpha_{i}^{(1)}(0) = 0\). The purpose now is to find out how many terms are in the expressions \(\alpha_{i}^{(1)}(k), k = 0, 1, \ldots\) for each \(i\) retrieved so far. From

\[
\text{rank} \begin{pmatrix} \alpha_{i}^{(1)}(0) & \alpha_{i}^{(1)}(1) \\ \alpha_{i}^{(1)}(1) & \alpha_{i}^{(1)}(2) \end{pmatrix} = 1, \quad i = 3, 4,
\]

we know that there is no collision of terms for \(i = 3, 4\). From

\[
\text{rank} \begin{pmatrix} \alpha_{i}^{(1)}(0) & \alpha_{i}^{(1)}(1) & \alpha_{i}^{(1)}(2) \\ \alpha_{i}^{(1)}(1) & \alpha_{i}^{(1)}(2) & \alpha_{i}^{(1)}(3) \\ \alpha_{i}^{(1)}(2) & \alpha_{i}^{(1)}(3) & \alpha_{i}^{(1)}(4) \end{pmatrix} = 2, \quad i = 1
\]
we know that $\alpha_n^{(1)}(k)$ consists of 2 terms, which can be retrieved as illustrated in Section 5.2.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\alpha_i$</th>
<th>$\phi_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$6.5 \exp(0.15i)$</td>
<td>$-0.19 - i2\pi 453.1$</td>
</tr>
<tr>
<td>2</td>
<td>6.8</td>
<td>$-0.132 - i2\pi 452.19$</td>
</tr>
<tr>
<td>3</td>
<td>$6.8 \exp(0.3i)$</td>
<td>$-0.183 - i2\pi 451.02$</td>
</tr>
<tr>
<td>4</td>
<td>$6.4 \exp(0.9i)$</td>
<td>$-0.11 - i2\pi 450.21$</td>
</tr>
<tr>
<td>5</td>
<td>$7.1 \exp(0.7i)$</td>
<td>$-0.21 - i2\pi 448.39$</td>
</tr>
<tr>
<td>6</td>
<td>$4.71 \exp(0.12i)$</td>
<td>$-0.106 - i2\pi 132.5$</td>
</tr>
<tr>
<td>7</td>
<td>$3.9 \exp(0.1i)$</td>
<td>$-0.129 - i2\pi 131.4$</td>
</tr>
<tr>
<td>8</td>
<td>$7.2 \exp(-0.234i)$</td>
<td>$-0.203 - i2\pi 130.01$</td>
</tr>
<tr>
<td>9</td>
<td>$7.43 \exp(0.2i)$</td>
<td>$-0.16 - i2\pi 129.17$</td>
</tr>
<tr>
<td>10</td>
<td>$4.4 \exp(-0.52i)$</td>
<td>$-0.19 - i2\pi 128.39$</td>
</tr>
<tr>
<td>11</td>
<td>$3 \exp(0.21i)$</td>
<td>$-0.101 + i2\pi 9.1$</td>
</tr>
<tr>
<td>12</td>
<td>$3 \exp(-0.8i)$</td>
<td>$-0.127 + i2\pi 11.81$</td>
</tr>
<tr>
<td>13</td>
<td>$7.2 \exp(-0.106i)$</td>
<td>$-0.21 + i2\pi 126.01$</td>
</tr>
<tr>
<td>14</td>
<td>$6.53 \exp(0.2i)$</td>
<td>$-0.15 + i2\pi 127.62$</td>
</tr>
<tr>
<td>15</td>
<td>$6.7 \exp(-0.3i)$</td>
<td>$-0.173 + i2\pi 128.98$</td>
</tr>
<tr>
<td>16</td>
<td>$6.8 \exp(-0.15i)$</td>
<td>$-0.11 + i2\pi 334.01$</td>
</tr>
<tr>
<td>17</td>
<td>$6 \exp(0.26i)$</td>
<td>$-0.12 + i2\pi 335.18$</td>
</tr>
<tr>
<td>18</td>
<td>$7.1 \exp(-0.2i)$</td>
<td>$-0.157 + i2\pi 336.01$</td>
</tr>
<tr>
<td>19</td>
<td>7.1</td>
<td>$-0.120 + i2\pi 337.91$</td>
</tr>
<tr>
<td>20</td>
<td>$6 \exp(-0.1i)$</td>
<td>$-0.18 + i2\pi 339.61$</td>
</tr>
</tbody>
</table>

Table 1: Collision-free example.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\alpha_i$</th>
<th>$\phi_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>18</td>
<td>$i2\pi 191.9$</td>
</tr>
<tr>
<td>2</td>
<td>$-20$</td>
<td>$i2\pi 291.9$</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>$i2\pi 391.9$</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>$i2\pi 526.2$</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>$i2\pi 858.1$</td>
</tr>
<tr>
<td>6</td>
<td>11</td>
<td>$i2\pi 958.1$</td>
</tr>
</tbody>
</table>

Table 2: Example where collisions occur.
Acknowledgements
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REFERENCES


