Region and Contour Identification of Physical Objects

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Received 1 December 2004, revised 6 December 2004, accepted 6 December 2004
Published online 20 December 2004

The region occupied by and the contour of a physical object in 3-dimensional space are in a way dual or interchangeable characteristics of the object: the contour is the region’s boundary and the region is contained inside the contour. In the same way the characterization of the object’s contour by its Fourier descriptors, and the reconstruction of its region from the object’s multidimensional moments, are also dual problems. While both problems are well-understood in two dimensions, the complexity increases tremendously when moving to the three-dimensional world.

In Section 2 we discuss how the latest techniques allow to reconstruct an object’s shape from the knowledge of its moments. For 2D significantly different techniques must be used, compared to the general 3D case.

In Section 3, the parameterization of a 2D contour onto a unit circle and a 3D surface onto a unit sphere is described. Furthermore, the theory of Fourier descriptors for 2D shape representation and the extension to 3D shape analysis are discussed.

The reader familiar with the use of either Fourier descriptors or moments as shape descriptors of physical objects may find the comparative discussion in the concluding section interesting.
In many instances of shape reconstruction, indirect object measurements of the underlying shape can be distilled into moments. Such inverse moment problems appear in diverse areas, among which probability and statistics [14], signal processing [43], heat conduction [1], computed tomography [35, 36] and magnetic and gravitational anomaly detection [8, 47]. Besides in shape reconstruction, moment based techniques have also extensively been used in imagery for image analysis and object representation or recognition [25, 33, 32, 37, 40, 49, 48, 56]. For such applications, the invariance to translation, rotation and scaling is essential.

Both Fourier descriptors and multidimensional moments can be made invariant to translation, rotation and scaling of the object under investigation. For 2D Fourier descriptors this is achieved by dividing all Fourier coefficients by the Fourier coefficient indexed by zero and ignoring the phases. Moment invariants insensitive to the same transformations are obtained by first computing their central and aspect ratio invariant form and then combining these into rotation invariant expressions. The transformation invariance of these parameters, extracted either from the object’s contour or its overall region, makes them tractable shape descriptors. However, for the applications under investigation in this paper, the transformation invariance is not required.

In this paper, the latest techniques to reconstruct an object’s shape from the knowledge of its moments is discussed and the theory of Fourier descriptors for shape representation is reviewed. For both topics the non-trivial extension from 2D to 3D shape analysis is discussed.

2 Moment information

Let us first describe how the problem of identifying a shape from its moments is dealt with in two dimensions. Since both variables are real, the problem can also be viewed as a one-dimensional problem in a complex variable. Therefore, the complexity of the two-dimensional formulation is not representative for that of the higher dimensional problem in general.

2.1 2D reconstruction

In case the object is a polygon [18], or when it defines a quadrature domain in the complex plane [21], its shape can be reconstructed from the knowledge of its moments, using a connection between formally orthogonal polynomials and numerical quadrature. We summarize the polygonal case because it allows an exact reconstruction.

Let \( \{c_k\}_{k \in \mathbb{N}} \) denote the sequence of moments of the polygon \( P \subset \mathbb{C} \),

\[
c_k = \int \int_P (x + iy)^k \, dx \, dy
\]  

and let us define linear functionals \( c \) and \( c^{(n)} \) from the vector space \( \mathbb{C}[z] \) to \( \mathbb{C} \) by

\[
c(z^k) = c_k \quad k \geq 0
\]

\[
c^{(n)}(z^k) = c_{k+n} \quad n \geq 0, k \geq 0
\]

In the sequel we set \( c_k = 0 \) when \( k < 0 \). With the sequence \( \{c_k\}_{k \in \mathbb{N}} \) we also associate the Hankel matrices

\[
H^{(n)}_m = \begin{pmatrix}
c_n & \cdots & c_{n+m-1} \\
\vdots & \ddots & \vdots \\
c_{n+m-1} & \cdots & c_{n+2m-2}
\end{pmatrix} \quad H^{(n)}_0 = 1
\]  

and

\[
H^{(n)}_m(z) = \begin{pmatrix}
c_n & \cdots & c_{n+m-1} & c_{n+m} \\
\vdots & \ddots & \vdots & \vdots \\
c_{n+m-1} & \cdots & z^{m-1} & c_{n+2m-1} \\
1 & \cdots & z^{m-1} & z^m
\end{pmatrix} \quad H^{(n)}_0(z) = 1
\]
and the Hadamard polynomials
\[ p_m^{(n)}(z) = \frac{\det H_m^{(n)}(z)}{\det H_m^{(n)}} \quad m \geq 0, n \geq 0 \] (4)

These monic polynomials of degree \( m \) are formally orthogonal with respect to the linear functional \( c^{(n)} \) because they satisfy [7, pp. 40–41]
\[ c^{(n)}(z^i p_m^{(n)}(z)) = 0 \quad i = 0, \ldots, m - 1 \]

The functional \( c \) is called \( m \)-normal if
\[ \det H_i^{(n)} \neq 0 \quad n \geq 0 \quad i = 0, \ldots, m \]

If the moments \( c_k \) are not given by (1), but by
\[ c_k = k(k-1) \int \int_P (x+iy)^{k-2} \, dx \, dy \] (5)

then the zeroes of the Hadamard polynomial \( p_m^{(0)}(z) \) are precisely the \( m \) vertices of the polygon \( P \) [18]. This property is based on the following quadrature result from [12, 13].

**Theorem 2.1** Let \( z_1, \ldots, z_m \) denote the vertices of a polygon \( P \) in the complex plane and let \( f(z) \) be a function analytic in the closure of \( P \). Then there exist constants \( a_1, \ldots, a_m \) depending upon \( z_1, \ldots, z_m \) but independent of \( f \), such that
\[ \int \int_P f''(z) \, dx \, dy = \sum_{j=1}^m a_j f(z_j) \] (6)

If the region \( P \) is unknown but its complex moments \( c_k \) are known from (5), then (6) with \( f(z) = z^k \) for \( k \geq 2 \), can be seen as a means to determine the \( z_j \) and \( a_j \) that characterize \( P \).

The zeroes of \( p_m^{(0)}(z) \) can be obtained by solving the generalized eigenvalue problem
\[ H_m^{(1)} u = z H_m^{(0)} u \]

where \( u \) denotes a vector in \( \mathbb{C}^m \). The solution of this generalized eigenvalue problem can be obtained most stably by the QZ algorithm, which can be improved as in [18] because of the special form of \( H_m^{(0)} \) and \( H_m^{(1)} \). The replication of columns of \( H_m^{(0)} \) in \( H_m^{(1)} \) allows to compute the QR factorization of \( H_m^{(0)} \) augmented by the last column of \( H_m^{(1)} \). This brings \( H_m^{(0)} \) in triangular form and \( H_m^{(1)} \) in upper Hessenberg form, as required.

After determining the vertices \( z_j \), the coefficients \( a_j \) in the quadrature result can be computed from
\[
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1 \\
\end{pmatrix}
\begin{pmatrix}
a_1 \\
a_2 \\
\vdots \\
a_m \\
\end{pmatrix} =
\begin{pmatrix}
c_0 \\
c_1 \\
\vdots \\
c_{m-1} \\
\end{pmatrix}
\]

Then the interior of the polygon has to be reconstructed. This is important in case the polygon is not assumed to be convex [36]. In [18] it is indicated how the conditioning can be improved by using so-called transformed moments with respect to a shifted and scaled basis. For a numerical illustration of the above technique we refer to [18].

### 2.2 3D reconstruction

Since an analogue of Theorem 2.1 does not exist in higher dimensions, another approach must be followed in three real dimensions. The Hadamard polynomial \( p_m^{(n)}(z) \) is also connected to Padé approximation theory by the following. Given a series of the form
\[
g(z) = \sum_{i=0}^{\infty} (-1)^i c_i z^i \] (7)
the Padé approximant $r_{m-1,m}(z) = p_{m-1,m}(z)/q_{m-1,m}(z)$ to $g(z)$ of degree $m - 1$ in the numerator and $m$ in the denominator is computed from the conditions

$$p_{m-1,m}(z) = \sum_{i=0}^{m-1} a_i z^i \quad q_{m-1,m}(z) = \sum_{i=0}^{m} b_i z^i$$

(8)

$$\left( \sum_{i=0}^{\infty} (-1)^i c_i z^i \right) q_{m-1,m}(z) - p_{m-1,m}(z) = \sum_{i=0}^{\infty} d_i z^{2m+i}$$

(9)

An explicit formula for the Padé denominator $q_{m-1,m}(z)$ is given by

$$q_{m-1,m}(z) = z^m \frac{p_m(1/z)}{p_m^{-1}(1/z)}$$

Now let $g(z)$ be a Markov function, meaning that $g(z)$ is defined to be a function with the integral representation

$$g(z) = \int_a^b \frac{f(u)}{1+zu} \, du \quad z \not\in (-\infty, -1/b] \cup [-1/a, +\infty[\]

(10)

$$-\infty < a \leq 0 < b < +\infty$$

(11)

where $f(u) \geq 0$, and the coefficients $c_i$ are the moments of $f$

$$c_i = \int_a^b f(u) \, u^i \, du$$

(12)

Then the following convergence result for Padé approximants holds [3, p. 228].

**Theorem 2.2** The sequence $\{r_{m-1,m}(z)\}_{m \in \mathbb{N}}$ of Padé approximants to the Markov function (10) converges to (10) for $z \not\in (-\infty, -1/b] \cup [-1/a, +\infty[$. The rate of convergence is governed by

$$\lim_{m \to \infty} \sup |g(z) - r_{m+k,m}(z)|^{1/m} \leq \frac{\sqrt{1/z + b} - \sqrt{1/z + a}}{\sqrt{1/z + b} + \sqrt{1/z + a}}$$

Fortunately a multivariate generalization of the concept of Padé approximant exists that allows a convergence result like the one in Theorem 2.2. A trivariate Stieltjes function $g(v, w, z)$ is defined by the integral representation

$$g(v, w, z) = \int_0^\infty \int_0^\infty \int_0^\infty \frac{f(t, s, u)}{1+(vt+us+zu)} \, dt \, ds \, du$$

(13)

with $f(t, s, u) \geq 0$ and finite real-valued moments

$$c_{ijk} = \int_0^\infty \int_0^\infty \int_0^\infty t^i s^j u^k f(t, s, u) \, dt \, ds \, du$$

A formal expansion of (13) provides a trivariate Stieltjes series of the form

$$g(v, w, z) = \sum_{i,j,k=0}^{\infty} (-1)^i+j+k \binom{i+j+k}{i+j} \binom{i+j}{i} c_{ijk} v^i w^j z^k$$

(14)

Given (14) one can compute the $(m-1,m)$ homogeneous trivariate Padé approximant to (14) as follows. First, we introduce the homogeneous expressions

$$A_{\ell}(v, w, z) = \sum_{i+j+k=\ell} a_{ijk} v^i w^j z^k$$

$$B_{\ell}(v, w, z) = \sum_{i+j+k=\ell} b_{ijk} v^i w^j z^k$$
and define the polynomials
\[ p_{m-1,m}(v, w, z) = \sum_{\ell=(m-1)m}^{(m-1)m+m-1} A_\ell(v, w, z) \]
\[ q_{m-1,m}(v, w, z) = \sum_{\ell=(m-1)m}^{(m-1)m+m} B_\ell(v, w, z) \]

Then, with
\[ C_\ell(v, w, z) = \sum_{i+j+k=\ell} (i+j+k) \binom{i+j+k}{i} c_{ijk} v^i w^j z^k \]  \quad (15)

we write down the homogeneous accuracy-through-order conditions
\[ \left( \left( \sum_{\ell=0}^{\infty} (-1)^\ell C_\ell \right) q_{m-1,m} - p_{m-1,m} \right)(v, w, z) = \sum_{i+j+k=m(m-1)+2m}^{\infty} d_{ijk} v^i w^j z^k \]  \quad (16)

It has been shown [10, pp. 60–61] that a nontrivial solution for \( p_{m-1,m}(v, w, z) \) and \( q_{m-1,m}(v, w, z) \) can always be computed from (16). Moreover, all solutions \( p_{m-1,m}(v, w, z)/q_{m-1,m}(v, w, z) \) deliver the same unique irreducible form \( r_{m-1,m}(v, w, z) \) which is called the homogeneous Padé approximant to (14). This multivariate generalization of the concept of Padé approximant is the one most closely related to the univariate Padé approximant because of the following projection property. Let, for particular \(-\pi \leq \theta \leq \pi\) and \(0 \leq \phi \leq \pi/2\), the one-dimensional subspace \( S_{\theta, \phi} \subset \mathbb{R}^3 \) be given by
\[ S_{\theta, \phi} = \{ (z \cos \phi \cos \theta, z \cos \phi \sin \theta, z \sin \phi) \mid z \in \mathbb{R} \} \]

and denote the restriction of \( r_{m-1,m}(v, w, z) \) to \( S_{\theta, \phi} \) by \( r_{(0, \phi)}^{m-1,m}(z) \). When projecting the homogeneous polynomials \( C_\ell(v, w, z) \) given by (15) on \( S_{\theta, \phi} \), we find
\[ C_\ell(z \cos \phi \cos \theta, z \cos \phi \sin \theta, z \sin \phi) = \sum_{i+j+k=\ell} (i+j+k) \binom{i+j+k}{i} c_{ijk}(\cos \phi \cos \theta)^i(\cos \phi \sin \theta)^j(\sin \phi)^k z^\ell \]
\[ = C_\ell(\cos \phi \cos \theta, \cos \phi \sin \theta, \sin \phi) z^\ell \]

Replacing \( c_i \) in (4) by \( C_i(\cos \phi \cos \theta, \cos \phi \sin \theta, \sin \phi) \), parameterizes the Hadamard polynomial \( p_{m-1}^{(m-1)}(z) \) by \( \theta \) and \( \phi \). We however don’t want to burden the notation for \( p_{m-1}^{(m-1)}(z) \) with two additional parameters. An explicit formula for the Padé denominator \( q_{m-1,m}(v, w, z) \) restricted to \( S_{\theta, \phi} \) can be given in terms of this parameterized Hadamard polynomial:
\[ q_{m-1,m}(z \cos \phi \cos \theta, z \cos \phi \sin \theta, z \sin \phi)/z^{m(m-1)} = z^{m} p_{m-1}^{(m-1)}(1/z) \]

A more general determinant representation also holds [10, p. 17]. The above projection property makes it possible to compute the overall approximant \( r_{m-1,m}(v, w, z) \) for \( g(t, v, w, z) \) and subsequently apply Theorem 2.2 on every slice \( S_{\theta, \phi} \) [11]. When \( f(t, s, u) \) in (13) is the characteristic function of the object \( A \), appropriately scaled, and a number of moments \( c_{ijk} \) of \( A \) is given, then:

- the left-hand side of (13) can be obtained and evaluated at several \((v_\ell, w_\ell, z_\ell)\) in the unit ball, through its Padé approximant \( r_{m-1,m}(v, w, z) \) of sufficiently high degree;
- the right-hand side of (13) can be discretized using a cubature rule for the unit ball, which leads to a linear system in the unknowns \( f(t_h, s_h, u_h) \) where each \((t_h, s_h, u_h)\) is a node of the cubature rule;

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• the resulting linear system, which is ill-posed [22], can be regularized using the truncated SVD algorithm and then solved for the \( f(t_h, s_h, u_h) \).

Besides reconstructing three-dimensional objects, the above technique can also be used for the reconstruction of non-polygonal 2D shapes. Several examples of both 2D and 3D reconstructed objects can be found in [11].

3 Fourier descriptors

Fourier descriptors are widely used in shape description and recognition problems. An essential preprocessing step to be carried out before the actual computation of the Fourier coefficients and Fourier descriptors is the discrete parameterization of the given contour. In particular, a 2D contour must be mapped onto a circle, after which it can be expanded into elliptical harmonic functions. In 3D, a surface must be mapped onto a sphere, after which it can be expanded into spherical harmonic functions. As is explained in the next sections, the latter is all but trivial compared to the 2D analogon.

Fourier descriptors are particularly valuable for parameterizations that exhibit periodicity and that can be modelled by means of a limited number of Fourier functions.

3.1 2D parameterization

The boundary of a simply connected two-dimensional region is a closed contour. The simplest closed contour that shares the same topology as an arbitrary closed contour is the unit circle

\[
S^1 = \{ z \in \mathbb{C} : |z|^2 = 1 \}
\] (17)

and \( S^1 \) is therefore an adequate parameter space. The mapping of the 2D contour onto the unit circle is trivial. Indeed, if the contour has \( N \) pixels, then \( N \) points can be equally distributed over the unit circle. The locations of these points are given by \((\cos \phi_n, \sin \phi_n)\) with \( \phi_n = 2\pi n/N \). Hence, the polar coordinate \( \phi_n \) parameterizes the contour. By traversing the contour in object space, the unit circle is traversed as well at a constant speed. Each angle \( \phi_n \) corresponds to a unique point \((x_n, y_n)\) on the contour. Note that the contour is a periodic function of \( \phi_n \). Hence, the harmonic functions are the preferred set of basis functions.

It is important to note that the sampling of the unit circle must be uniform. If this uniformity is not met, the orthogonality relation of the Fourier basis functions evaluated in discrete points on the unit circle, does not hold, which results in erroneous Fourier descriptors.

3.2 3D parameterization

Similar to parameterization in 2D, a surface can be mapped onto a unit sphere. In 3D, however, no direction or order exists. During the past ten years, various methods were proposed to deal with this mapping problem. An interesting approach to map a polyhedron onto a sphere has been proposed in [4] where a continuous one-to-one mapping from the surface of the original object to the surface of a unit sphere is proposed. Thereby, two poles are selected on the surface, after which the vertices are uniformly mapped onto the sphere by means of a diffusion process. Because of this diffusion process, however, the algorithm may be a computational burden, especially for polygon models with a large number of vertices. The mapping procedure can be improved based on the use of progressive meshes [24, 44, 39, 19, 20].

A progressive mesh is a multi-resolution representation of a mesh in which edges are iteratively collapsed based on a chosen criterion [16]. In [17, 23] a fast and efficient method to obtain a multi-resolutional representation of a polyhedron using a quadric error metric is developed [17, 23]. During the simplification, vertex pairs are iteratively contracted such that this quadric error is minimal. Simplification of the polyhedron is continued until only a few vertices remain. Thereby, constraints are imposed on the simplification scheme so as to ensure that the topology is not changed. After the construction of the progressive mesh, the strongly simplified polyhedron is mapped onto the unit sphere, after which iterative reconstruction is performed on the sphere.

Mapping of 3D surfaces to a parameter domain is an essential preprocessing step. At the moment, efficient parameterization methods are available to map genus-0 objects onto a sphere. However, depending on the object’s shape, other parameterization domains may be more suitable. As indicated in [26], tubular objects are more
suited to be parameterized on a cylinder. Also, objects with a higher genus should be mapped onto the appropriate domain. For example, a genus-1 object (closed object with one hole) has a donut-like shape as its natural parameterization domain. Developing a suitable parameterization for each surface type is a challenge.

3.3 2D Fourier descriptors

A closed contour \( C \) in 2D can be represented by a complex function \( z(t) \), where \( z(t) = x(t) + iy(t) \). Thereby, \( x(t) \) and \( y(t) \) represent the \( x \)- and \( y \)-coordinates of the contour points. If \( z(t) \) is periodic with period \( T \), i.e. \( z(t) = z(t + kT) \), expansion of \( z(t) \) into a Fourier series yields

\[
z(t) = \sum_{n=-\infty}^{\infty} a_n \exp\left(\frac{2\pi i nt}{T}\right)
\]

where the Fourier coefficients are found from the Fourier transform of \( z(t) \):

\[
a_n = \int_0^T z(t) \exp\left(-\frac{2\pi i nt}{T}\right) dt
\]

If the contour is sampled on a discrete grid, the discrete Fourier transform of \( z(t) \) is given by

\[
a_n = \frac{1}{N} \sum_{t=0}^{N-1} z(t) \exp\left(-i\phi_n t\right) \quad \phi_n = \frac{2\pi n}{N}
\]

Subsequently, the Fourier coefficients \( a_n \) are appropriately transformed to make them independent to translation, rotation, scale, and starting point, by dividing all coefficients \( a_n \) by \( a_0 \) and ignoring the phases.

3.4 3D Fourier descriptors

Consider a 3D closed object, of which the surface is described by an object function \( r(x, y, z) \). A suitable parameterization domain for a closed surface is the unit sphere \( S^2 \), since they share the same topology. The parameterization defines the relation between the coordinates \( n(\theta, \phi) = (\cos \phi \cos \theta, \cos \phi \sin \theta, \sin \phi)^T \) of points located on the unit sphere and the coordinates on the object’s surface:

\[
S(\theta, \phi) = \begin{pmatrix}
x(\theta, \phi) \\
y(\theta, \phi) \\
z(\theta, \phi)
\end{pmatrix}
\]

Now assume that \( N \) observations are available at the locations \( n_i, i = 1, \ldots, N \). Then \( S(\theta, \phi) \) at a new location \( n(\theta, \phi) \) can be approximated from the observations \( S(\theta_i, \phi_i) \) using the spherical harmonic functions \( Y^m_\ell \):

\[
S(\theta, \phi) \approx \hat{S}(\theta, \phi) = \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} a_{\ell m} Y^m_\ell(\theta, \phi)
\]

where the Fourier coefficients \( a_{\ell m} \) are the inner product of the vector \( S \) with the spherical harmonic function \( Y^m_\ell \):

\[
a_{\ell m} = \langle S, Y^m_\ell \rangle = \int_0^{2\pi} \int_0^\pi S(\theta, \phi) Y^m_\ell(\theta, \phi) \sin \theta d\theta d\phi
\]

In practical applications, these coefficients are often estimated by a least squares fitting of the spherical harmonics to the discrete observations:

\[
\hat{\theta} = \sum_{i=1}^{N} S(\theta_i, \phi_i) y^H(\theta_i, \phi_i)(Y^H Y)^{-1}
\]
Fig. 1  a) original mesh  b) spherical mapping  c) limited Fourier reconstruction

where \( y(\theta, \phi) = [Y^0_0(\theta, \phi), Y^{-1}_1(\theta, \phi), Y^1_1(\theta, \phi), ..., Y^L_L(\theta, \phi)]^T \) and \( Y = [y_1(\theta_1, \phi_1), ..., y_N(\theta_N, \phi_N)]^T \).

Note that the above equation simplifies significantly in case the observations are regularly spaced since in that case the discrete version of the orthogonality conditions

\[
\int_0^{2\pi} \int_0^{\pi} Y^{m'*}(\theta, \phi) Y^m(\theta, \phi) \sin \theta d\theta d\phi = \delta_{l', l} \delta_{m', m}
\]

is satisfied, where \( Y^{m'*} \) denotes the complex conjugate of \( Y^m \).

By making the Fourier coefficients \( a^m_l \) invariant to translation, rotation, and scale [4], 3D Fourier descriptors are obtained. Similarly to 2D, translation invariance is achieved by ignoring the component \( a_0 \). Rotation invariance can be obtained by rotating the object to a standard orientation. Indeed, a representation of the object by only the spherical harmonics of degree 1 is an ellipsoid, which can be aligned along an axis of choice[5]. The same transformation parameters are then applied to the whole object. In case of degeneracy, e.g. when the ellipsoid is a sphere, spherical harmonics of higher degree can be taken into account. Finally, scale invariance can be achieved by dividing all descriptors by the length of the main axis of the ellipsoid.

The computation of 3D Fourier descriptors itself is not fundamentally different from the computation of 2D Fourier descriptors. However, the parameterization step preceding the computation of Fourier coefficients from surface data is significantly more difficult in 3D than in 2D.

Reconstruction from a limited number of Fourier coefficients allows a multi-resolution approach. In particular, reconstruction from a spherical harmonic with degree 0 yields the sphere that matches the object best in a least squares sense. Similarly, reconstruction from spherical harmonics with maximum order 1 finds the best fitting ellipsoid. The more coefficients are used for the reconstruction, the more detail is added to the reconstruction. An example of a reconstruction of a polyhedron with a limited number of spherical harmonics is shown in Fig. 1.

4 Conclusion

From the sections 2.1, 3.1 and 3.3 the following interplay between Fourier descriptors and moments for closed 2D curves is apparent. Let \( z(t) = x(t) + iy(t) \) describe a closed 2D contour \( C \), bounding a compact 2D area. Let \( N \) points be sampled on the contour, thus approximating the curve by a polygonal contour with \( N \) vertices. On one hand, if a sufficient number of moments of the polygon are available, namely \( 2N - 1 \), Theorem 2.1 enables one to reconstruct the polygonal contour representing \( z(t) \). On the other hand, the Fourier coefficients (20) characterize \( z(t) \) and can further be used for a variety of applications, among which smoothing, classification and the like.

While the polygonal/polyhedral representation of a physical object continues to play a major role in the computation of 3D Fourier descriptors, this stepping stone can be omitted in 3D shape reconstruction from moments. However, this kind of generality comes at a price. At this moment, it is not possible to reconstruct only the boundary of a 3D object from moment information. One reconstructs the entire area occupied by the object. It is our hope that a better understanding of the interplay between

- techniques for the 3D parameterization of several kinds of objects, and
- formulas expressing moments in terms of shape characteristics such as vertices,

can lead to the removal of this drawback.
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