On the continuity of the multivariate Padé operator

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Abstract: Continuity of the univariate Padé operator was proved in [5,6]. We discuss the limitations of a multivariate generalization and prove a multivariate analogon of the continuity property.

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1. The multivariate Padé operator

Let \( f(x_1, \ldots, x_p) \) be a multivariate function given by its Taylor series expansion

\[
f(x_1, \ldots, x_p) = \sum_{k=0}^{\infty} \sum_{k_1 + \cdots + k_p = k} c_{k_1 \ldots k_p} x_1^{k_1} \cdots x_p^{k_p}.
\]

Choose \( n \) and \( m \) in \( \mathbb{N} \) and define multivariate polynomials

\[
p(x_1, \ldots, x_p) = \sum_{i=nm}^{nm+n} \sum_{i_1 + \cdots + i_p = i} a_{i_1 \ldots i_p} x_1^{i_1} \cdots x_p^{i_p}
\]

and

\[
q(x_1, \ldots, x_p) = \sum_{j=nm}^{nm+m} \sum_{j_1 + \cdots + j_p = j} b_{j_1 \ldots j_p} x_1^{j_1} \cdots x_p^{j_p}.
\]

In the multivariate Padé approximation problem, defined in [2], we calculate the coefficients \( a_{i_1 \ldots i_p} \) and \( b_{j_1 \ldots j_p} \) such that

\[
(f \cdot q - p)(x_1, \ldots, x_p) = \sum_{k=nm+n+1}^{\infty} \sum_{k_1 + \cdots + k_p = k} d_{k_1 \ldots k_p} x_1^{k_1} \cdots x_p^{k_p},
\]

where \( d_{k_1 \ldots k_p} \) are the coefficients of the approximant.

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It is always possible to compute a nontrivial solution of (1) and different solutions supply equivalent rational functions $p/q$. For more details we refer to [1].

The irreducible form $(p_*/q_*)(x_1, \ldots, x_p)$ of $(p/q)(x_1, \ldots, x_p)$ is called the $(n, m)$ multivariate Padé approximant for $f$ and it is unique up to a multiplicative constant.

In [1] we proved the following results for $p_*/q_*$. Let $\delta_0 q_*$ denote the order of the multivariate polynomial $q_*$, in other words the degree of the first nonzero term in $q_*(x_1, \ldots, x_p)$; let $\delta p_*$ and $\delta q_*$ respectively denote the exact degree of the multivariate polynomials $p_*$ and $q_*$. Then it is easy to see that $\delta_0 p_* > \delta_0 q_*$ and that for

$$n' = \delta p_* - \delta_0 q_*, \quad m' = \delta q_* - \delta_0 q_*,$$

we have

(a) $n' \leq n$,
(b) $m' \leq m$,
(c) there exists an $s$ in $\mathbb{N}$, $0 \leq s \leq \min(n - n', m - m')$, such that for $s = nm - \delta_0 q_* + s$, we can find a nontrivial

$$r(x_1, \ldots, x_p) = \sum_{k_1 + \ldots + k_p = s} e_{k_1 \ldots k_p} x_1^{k_1} \ldots x_p^{k_p}$$

with $\delta_0 [(f \cdot q_* - p_*) \cdot r] \geq nm + n + m + 1$,
(d) $\delta_0 (f \cdot q_* - p_*) = \delta_0 q_* + n' + m' + t + 1$ with $t \geq \max(n - n', m - m')$.

In other words, the polynomials $p_*$ and $q_*$ themselves do not necessarily satisfy (1) anymore, but we can multiply them by an appropriate homogenous polynomial $r$ to obtain a solution of (1).

Let us denote $\min(n - n', m - m')$ by $d_{n,m}$ and call it the defect of $(p_*/q_*)(x_1, \ldots, x_p)$. This terminology is chosen for $d_{n,m}$ because one can see from (2c) that

$$\delta_0 (f \cdot q_* - p_*) \geq \delta_0 q_* + n + m + 1 - d_{n,m}.$$

Let $T_{n,m}$ be the operator which associates with $f(x_1, \ldots, x_p)$ the $(n, m)$ multivariate Padé approximant; then $T_{n,m}$ is called the multivariate Padé operator.

2. Continuity

In the Taylor series expansion of $f(x_1, \ldots, x_p)$, the contribution

$$\sum_{k_1 + \ldots + k_p = k} c_{k_1 \ldots k_p} x_1^{k_1} \ldots x_p^{k_p}$$

is the result of a $k$-linear operator on $\mathbb{R}^p$ [4, pp. 100–107]. If we call that $k$-linear operator $C_k$ and write $x = (x_1, \ldots, x_p)$, then we have

$$f(x_1, \ldots, x_p) = \sum_{k=0}^{\infty} C_k x^k,$$

where $C_k x^k = C_k x \cdots x$. $k$ times

Analogously we can write

$$p(x) = \sum_{i=nm}^{nm+m} A_i x^i, \quad q(x) = \sum_{j=nm}^{nm+m} B_j x^j.$$
We can now introduce seminorms for the power series $f$ as follows:

$$
||f(x_1, \ldots, x_p)||_I = \max_{0 \leq k \leq l} ||C_k||,
$$

where $||C_k|| = \max_{x \in I} |C_k x^k|$.

Let $x_0 = (x_{01}, \ldots, x_{0p})$ be such that $q(x_0) \neq 0$. We do not necessarily have $x_0 = 0$ because $\partial_0 q$ may be strictly positive. Then, because of the continuity of $q$, there is a finite poly-interval $I_1 \times \cdots \times I_p$ around $x_0$, where $q$ is nonzero. We call this poly-interval $I$. Multivariate functions $g$, continuous on $I$, are normed by the Chebyshev norm

$$
||g||_{\infty} = \max_{x \in I} |g(x_1, \ldots, x_p)|.
$$

Continuity of the univariate Padé operator was extensively discussed in [6,5]. We shall now attempt to prove a multivariate analogon of those conclusions. But first of all we want to show why we cannot expect a continuity property of this multivariate Padé operator $T_{n,m}$ to hold in the origin $x_0 = 0$.

Consider

$$
f(x_1, x_2) = \frac{1}{1 - x_1} = 1 + x_1 + x_1^2 + x_1^3 + \cdots
$$

and

$$
\hat{f}(x_1, x_2) = f(x_1, x_2) + \alpha x_2^2
$$

$$
= 1 + x_1 + x_1^2 + \alpha x_2^2 + x_1^3 + \cdots.
$$

Then

$$
\lim_{\alpha \to 0} ||\hat{f} - f||_{n+m} = 0.
$$

In other words, $\hat{f}$ can be chosen as close to $f$ as we want.

Take $n = 1 = m$ and calculate the $(n, m)$ Padé approximants for $f$ and $\hat{f}$. We get

$$
\frac{p_*(x_1, x_2)}{q_*(x_1, x_2)} = \frac{1}{1 - x_1} \quad \text{for } f \quad \text{and} \quad \frac{\hat{p}_*(x_1, x_2)}{\hat{q}_*(x_1, x_2)} = \frac{x_1 - \alpha x_2^2}{x_1 - x_1^2 - \alpha x_2^2} \quad \text{for } \hat{f}.
$$

In both cases the denominator polynomials $q(x_1, x_2)$ and $\hat{q}(x_1, x_2)$ of all solutions of (1) are zero in the origin because $nm > 0$.

The order $\partial_0 q$ or $\partial_0 \hat{q}$ and what is left of it in $q_*(x_1, x_2)$ or $\hat{q}_*(x_1, x_2)$ are responsible for the singularity in the origin and hence for

$$
||T_{n,m}f - T_{n,m}\hat{f}||_{\infty} = \infty
$$

on every poly-interval $I$ around the origin. Nevertheless, the following continuity property can be proved.

Let $f$ and $\hat{f}$ be multivariate power series and let $n$ and $m$ be fixed.

**Theorem 2.1.** If $d_{n,m} = 0$ and $q(x) \neq 0$ for all $x$ in a suitably chosen poly-interval $I$, then

$$
\forall \epsilon, \exists \delta: ||(f - \hat{f})(x_1, \ldots, x_p)||_{n+m} < \delta \Rightarrow ||T_{n,m}f - T_{n,m}\hat{f}||_{\infty} < \epsilon.
$$
Proof. If \( d_{n,m} = 0 \) then \( \partial q = n m \) and \( A_n x^{n+1} + B_m x^{n+1} \neq 0 \) is not identically zero (\( A_n \) and \( B_m \) contain the terms of degree \( nm + n \) and \( nm + m \) in \( p(x) \) and \( q(x) \) respectively). In [3] we showed that if \( D_{n,m}(C) \neq 0 \), \( A_n \) and \( B_m \) can be given by
So because $D_{n,m}(C)$, $A_n x^{nm+n}$, $B_m x^{nm+m}$, are determinants and thus continuous functions of the $C_k$, we have for $C_k$ close to $C_k$

$$D_{n,m}(C) \neq 0, \quad |A_n x^{nm+n}| + |B_m x^{nm+m}|$$

does not vanish identically, where $A_n$ and $B_m$ are the result of replacing $C_k$ by $C_k$ in the determinants above.

Thus, if we solve the Padé approximation problem for $f(x) = \sum_{k=0}^{\infty} C_k x^k$ and $\tilde{f}(x) = \sum_{k=0}^{\infty} \overline{C}_k x^k$ with $n$ and $m$ fixed and $\overline{C}_k$ close to $C_k$, we get solutions $p_c(x)$, $q_c(x)$ and $p_{\overline{c}}(x)$, $q_{\overline{c}}(x)$ and we get irreducible rational forms $(p_c/q_c)(x)$ and $(p_{\overline{c}}/q_{\overline{c}})(x)$. We recall from [3] that if $D_{n,m}(C) \neq 0$, we also have the following determinantal formulas for $p_c$ and $q_c$:

$$p_c(x) = \frac{\sum_{k=0}^{n} C_k x^k}{\sum_{k=0}^{n-m} C_k x^k}$$

$$q_c(x) = \frac{1}{C_{n+m} x^{n+m}}$$

The fact that $q_c(x)$ is a continuous function of the $C_k$ implies the existence of a constant $\delta$ such that for $\tilde{f} = \sum_{k=0}^{\infty} \overline{C}_k x^k$ with $\|f - \tilde{f}\|_{n+m} \leq \delta$, also $q_{\overline{c}}(x) \neq 0$ for all $x$ in the poly-interval $I$. Hence

$$\left\| \frac{p_c - \overline{p}_c}{q_c - \overline{q}_c} \right\|_{\infty} = \max_{x \in I} \left\| \frac{p_c(x) - \overline{p}_c(x)}{q_c(x) - \overline{q}_c(x)} \right\|_{\infty}$$

$$= \left\| \frac{p_c - \overline{p}_c}{q_c - \overline{q}_c} + \frac{p_c}{q_c} - \frac{p_c}{\overline{q}_c} \right\|_{\infty}$$

$$\leq \left\| \frac{p_c - \overline{p}_c}{q_c - \overline{q}_c} \right\|_{\infty} + \left\| \frac{p_c}{q_c} - \frac{p_c}{\overline{q}_c} \right\|_{\infty} + \left\| \frac{\overline{p}_c}{\overline{q}_c} - \frac{p_c}{\overline{q}_c} \right\|_{\infty},$$
where

\[ \left\| \frac{P(x) - \bar{C}}{q(x) - \bar{C}} \right\|_\infty = 0 = \left\| \frac{P(x) - \bar{C}}{q(x) - \bar{C}} \right\|_\infty, \]

because \( p(x)q(x) = p(x)q(x) \) and \( p(x)q(x) = p(x)q(x). \)

Now

\[ |p(x) - q(x)| \leq \sum_{i=0}^{n} \| A_i - A_i \| \cdot \| x \|^n, \]

\[ |q(x) - q(x)| \leq \sum_{j=0}^{m} \| B_j - B_j \| \cdot \| x \|^m. \]

Since \( \| A_i - A_i \| \leq L \cdot \| f - \hat{f} \|_{n+m} \) and since \( I \) is a finite poly-interval, we can write

\[ \| p(x) - q(x) \| \leq M \| f - \hat{f} \|_{n+m}. \]

Analogously,

\[ \| q(x) - q(x) \| \leq M \| f - \hat{f} \|_{n+m}. \]

Since \( q(x) \neq 0 \) in \( I \), we get a constant \( E \) such that

\[ |q(x)| > E \quad \text{for all} \quad x \in I. \]

So

\[ \left\| \frac{p(x) - q(x)}{q(x) - \bar{C}} \right\|_\infty = \left\| \frac{(p(x) - q(x))(q(x) - \bar{C})}{q(x)q(x)} \right\|_\infty \leq \frac{1}{q(x)} \cdot \frac{1}{E} (\| q(x) \|_\infty + \| p(x) \|_\infty) M \| f - \hat{f} \|_{n+m} \leq K \cdot \| f - \hat{f} \|_{n+m}, \]

and this terminates the proof, for we already had

\[ \left\| T_{n,m}f - T_{n,m} \hat{f} \right\|_\infty = \left\| \frac{p(x) - q(x)}{q(x) - \bar{C}} \right\|_\infty \leq \frac{p(x) - q(x)}{q(x) - \bar{C}}. \]

We have defined the defect \( d_{n,m} = \min(n - n', m - m') \), when \( (p(x)/q(x)) \) is the \((n, m)\) multivariate Padé approximant for \( f(x_1, \ldots, x_p) \) and \( n' = \partial p(x) - \partial_0 p(x), m' = \partial q(x) - \partial_0 q(x) \). Let us now take \( \hat{f}(x_1, \ldots, x_p) \) close to \( f(x_1, \ldots, x_p) \), i.e., \( \| f - \hat{f} \|_{n+m} \) small, and denote the defect for the \((n, m)\) multivariate Padé approximant for \( f \) by \( \delta_{n,m} \). Then we can prove the following property.

**Corollary 2.2.** If \( d_{n,m} = 0 \) for \( f \), then \( \bar{d}_{n,m} = 0 \) for \( \hat{f} \) close to \( f \).

**Proof.** If \( d_{n,m} = 0 \) then \( D_{n,m}(\bar{C}) \) is nontrivial and thus \( D_{n,m}(\bar{C})(x) \neq 0 \) is a dense set in \( \mathbb{R}^p \) because \( D_{n,m}(\bar{C}) \) is a polynomial in \( x \). Take \( \bar{x} \in D \). Then the system

\[ c_{n+1}b_0 + \cdots + c_{n+1-m}b_m = 0, \]

\[ \vdots \]

\[ c_{n+m}b_0 + \cdots + c_n b_m = 0, \]
where \( c_k = \bar{c}_k \bar{x}^k \) and where \( b_0, \ldots, b_m \) are unknown, has for \( b_0 = D_{n,m}(\bar{C})(\bar{x}) \) a unique solution \( b_1, \ldots, b_m \) where \( b_j \) is the result of an \((nm+j)\)-linear operator evaluated at \( \bar{x}^{nm+j} \), because

\[
\begin{vmatrix}
 c_n & -c_{n+1} & c_{n+1-m} \\
 \vdots & \ddots & \vdots \\
 c_{n+m-1} & -c_{n+m} & c_n \\
\end{vmatrix}
\]

\( j \)th column in \( D_{n,m}(\bar{C})(\bar{x}) \)

replaced by this column

\[
\begin{vmatrix}
 \bar{c}_n \bar{x}^n & -\bar{c}_{n+1} \bar{x}^{n+1} & \bar{c}_{n+1-m} \bar{x}^{n+1-m} \\
 \vdots & \ddots & \vdots \\
 \bar{c}_{n+m-1} \bar{x}^{n+m-1} & -\bar{c}_{n+m} \bar{x}^{n+m} & \bar{c}_n \bar{x}^n \\
\end{vmatrix}
\]

Let us denote this by \( b_j = \bar{B}_{nm+j} \bar{x}^{nm+j} \).

For \( x \) in \( \mathbb{R}^p \setminus D \), the value \( \bar{B}_{nm+j} \bar{x}^{nm+j} \) can uniquely be defined by continuity because \( D \) is dense. So there is only one solution

\[
\begin{align*}
\bar{p}(x_1, \ldots, x_p) &= \sum_{i=0}^{n} \bar{A}_{nm+i} x^{nm+i}, \\
\bar{q}(x_1, \ldots, x_p) &= \sum_{j=0}^{m} \bar{B}_{nm+j} x^{nm+j},
\end{align*}
\]

of the \((n, m)\) Padé approximation problem with \( \bar{B}_{nm} = D_{n,m}(\bar{C}) \).

Let \( \bar{p}_* \) and \( \bar{q}_* \) be numerator and denominator of the irreducible form of \( \bar{p}(x_1, \ldots, x_p)/\bar{q}(x_1, \ldots, x_p) \). Then from the polynomial \( u(x_1, \ldots, x_p) = \sum_{k=0}^{\delta u} U_k x^k \) such that

\[
\bar{p} = \bar{p}_* \cdot u, \quad \bar{q} = \bar{q}_* \cdot u,
\]

we have \( \delta_0 u = \partial u = nm - \delta_0 q_* \), otherwise

\[
\bar{p}_* \cdot U_{nm-\delta_0 q_*} \quad \text{and} \quad \bar{q}_* \cdot U_{nm-\delta_0 q_*}
\]

would be another solution of the \((n, m)\) Padé approximation problem with the same term of lowest degree in the denominator as \( \bar{p} \) and \( \bar{q} \).

So for \( \bar{C}_k \) close to \( C_k \) we have

\[
\begin{align*}
\partial \bar{p}_* - \partial_0 q_* &= (\partial \bar{p} - \partial u) - \partial_0 q_* = \partial \bar{p} - nm, \\
\partial \bar{q}_* - \partial_0 q_* &= (\partial \bar{q} - \partial u) - \partial_0 q_* = \partial \bar{q} - nm, \\
|\bar{A}_{nm+n} x^{nm+n}| + |\bar{B}_{nm+m} x^{nm+m}| \text{ nontrivial},
\end{align*}
\]

and thus

\[
\bar{d}_{n,m} = 0. \quad \Box
\]

The similarity of these results with the ones obtained for univariate Padé approximants is remarkable.
References