A COMPARISON OF SOME MULTIVARIATE PADÉ-APPROXIMANTS*

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Abstract. In [5] Levin defined general order Padé-type rational approximants of a function of n variables (here referred to as “type L” approximants). In §1 we repeat briefly the defining equations and the determinant representation for n=2. Levin proved that the Chisholm approximants were a special case of his “type L” approximants.

The multivariate Padé-approximants (here referred to as “type C” approximants) were introduced in [1] and [2]; we repeat the definition for n=2 in §2. They have several nice properties which often imply numerical advantages; examples of such situations are given in [3] and [4].

In §3 we show that “type C” approximants are also a special case of the “type L” approximants. The explicit determinant formulas are a link between the solution of the Padé approximation problem and the irreducible rational form of the solution. Via the determinant representation we can also see that, in the case of “type C” approximants, we deal with matrices that are near-Toeplitz. This is not true for the Chisholm approximants. A theorem concerning the displacement-rank of the matrix of the homogeneous system, defining the coefficients of the denominator of the “type C” approximant, is proved.

In §4 analogous results are formulated for n>2.

1. General order Padé-type rational approximants in two variables (or type L approximants). We repeat some notations and definitions given by Levin.

Let \( \mathbb{N} = \{0, 1, 2, \ldots\} \). Given a subset \( D \) of \( \mathbb{Z}^2 \) we define:

(a) the complement \( \overline{D} = \mathbb{Z}^2 \setminus D \),
(b) the \((i,j)\)-translation of \( D \) as \( D_{ij} = \{(k,n) | (k+i,n+j) \in D\} \),
(c) the nonnegative part of \( D \) as \( D^+ = D \cap \mathbb{N}^2 \).

To any subset \( D \) such that \( D^+ \) is a finite set we associate polynomials

\[
\sum_{(i,j) \in D^+} b_{ij} x^i y^j \quad \text{with} \quad b_{ij} \in \mathbb{R}
\]

We call \( D \) the rank of the polynomials. Given the double power series

\[
f(x,y) = \sum_{i,j=0}^{\infty} c_{ij} x^i y^j
\]

we will choose three subsets \( N, D \) and \( E \) of \( \mathbb{Z}^2 \) and construct an \([N/D]_E\) approximation to \( f(x,y) \) as follows:

\[
\begin{align*}
(1.1a) \quad P(x,y) &= \sum_{(i,j) \in N^+} a_{ij} x^i y^j \quad (N \leftarrow \text{numerator}), \\
(1.1b) \quad Q(x,y) &= \sum_{(i,j) \in D^+} b_{ij} x^i y^j \quad (D \leftarrow \text{denominator}), \\
(1.1c) \quad (f \cdot Q - P)(x,y) &= \sum_{(i,j) \in E^+} d_{ij} x^i y^j \quad (E \leftarrow \text{equations}).
\end{align*}
\]

We select \( N, D \) and \( E \) such that

(a) \( D \subseteq \mathbb{N}^2 \) has \( m \) elements, numbered

\[(i_1,j_1), \ldots, (i_m,j_m),\]

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(b) \( N \subseteq E \) and \( H = E \setminus N \) has \( m-1 \) elements in \( \mathbb{N}^2 \), numbered
\[(h_2,k_2), \ldots, (h_m,k_m) \quad (H \text{ homogeneous equations}).\]

Then \( P(x,y) \) and \( Q(x,y) \), defined by (1.1c), are given by:

\[
P(x,y) = \begin{vmatrix}
x^{i_1}y^{j_1}N_{ij_1}(x,y) & x^{i_2}y^{j_2}N_{ij_2}(x,y) & \cdots & x^{i_m}y^{j_m}N_{ij_m}(x,y) \\
c_{h_2-i_1,k_2-j_1} & c_{h_2-i_2,k_2-j_2} & \cdots & c_{h_2-i_m,k_2-j_m} \\
c_{h_3-i_1,k_3-j_1} & c_{h_3-i_2,k_3-j_2} & \cdots & c_{h_3-i_m,k_3-j_m} \\
\vdots & \vdots & \ddots & \vdots \\
c_{h_m-i_1,k_m-j_1} & c_{h_m-i_2,k_m-j_2} & \cdots & c_{h_m-i_m,k_m-j_m}
\end{vmatrix}
\]

where

\[
N_{ij_1}(x,y) = \sum_{(i,j) \in \mathbb{N}^2_0} c_{ij}x^iy^j,
\]

and

\[
Q(x,y) = \begin{vmatrix}
x^{i_1}y^{j_1} & x^{i_2}y^{j_2} & \cdots & x^{i_m}y^{j_m} \\
c_{h_2-i_1,k_2-j_1} & c_{h_2-i_2,k_2-j_2} & \cdots & c_{h_2-i_m,k_2-j_m} \\
c_{h_3-i_1,k_3-j_1} & c_{h_3-i_2,k_3-j_2} & \cdots & c_{h_3-i_m,k_3-j_m} \\
\vdots & \vdots & \ddots & \vdots \\
c_{h_m-i_1,k_m-j_1} & c_{h_m-i_2,k_m-j_2} & \cdots & c_{h_m-i_m,k_m-j_m}
\end{vmatrix}
\]

2. Multivariate Padé-approximants for a double power series (or type C approximants). We define a polynomial of degree \( l \) in two variables as

\[
\sum_{i+j=0}^{l} a_{ij}x^iy^j.
\]

A term \( a_{ij}x^iy^j \) is said to be of degree \( i+j \). The order \( \partial_0 P \) and the exact degree \( \partial P \) are defined by

\[
\partial_0 P = \min\{i+j | a_{ij} \neq 0\}, \quad \partial P = \max\{i+j | a_{ij} \neq 0\}.
\]

In the Padé-approximation problem of order \((l,m)\) we try to find a pair \((P,Q)\) of two-variable polynomials,

\[
(2.1a) \quad P(x,y) = \sum_{i+j=lm} a_{ij}x^iy^j,
\]

\[
(2.1b) \quad Q(x,y) = \sum_{i+j=lm} b_{ij}x^iy^j,
\]

such that

\[
(2.1c) \quad (f \cdot Q - P)(x,y) = \sum_{i+j=lm+l+m+1} d_{ij}x^iy^j.
\]

Equation (2.1c) is equivalent with

\[
\partial_0 (f \cdot Q - P) \geq lm + l + m + 1.
\]
A nontrivial $Q(x,y)$, such that \((2.1)\) is satisfied, always exists [2]. If the polynomials $R(x,y)$ and $S(x,y)$ also satisfy \((2.1)\), in other words if $\delta_0(f \cdot S - R) \geq lm + l + m + 1$, too, then

$$P(x,y) \cdot S(x,y) = Q(x,y) \cdot R(x,y).$$

This property justifies the following definitions:

(a) Let $(P_*/Q_*)(x,y)$ be the irreducible form of $(P/Q)(x,y)$ such that $Q_*(0,0) = 1$; if this form exists we call it the multivariate Padé-approximant of order $(l,m)$ for $f$.

(b) If the irreducible form $(P_*/Q_*)(x,y)$ is such that $\delta_0 Q_* \geq 1$, then we call $P_*/Q_*$ the multivariate rational approximant of order $(l,m)$ for $f$.

The $(l,m)$-multivariate rational approximant is unique up to a multiplicative constant in numerator and denominator. For $P_*(x,y)$ and $Q_*(x,y)$ we define:

$$l' = \delta P_* - \delta_0 Q_*, \quad m' = \delta Q_* - \delta_0 Q_*.$$

We can prove that

$$l' \leq l, \quad m' \leq m.$$

It is also easy to verify the following theorem.

**Theorem 2.1.** For the irreducible form $P_*/Q_*$ of $P/Q$ where $(P,Q)$ satisfies \((2.1)\) and for every polynomial $R(x,y) = \Sigma_i=0 r_i x^i y^{s-i}$ with $s = lm - \delta_0 Q_* + \min(l-l', m-m')$, $(P_*/R, Q_*)$ satisfies \((2.1)\).

Also $s = lm - \delta_0 Q_* + \min(l-l', m-m')$ is the highest possible degree that allows the construction of a homogeneous polynomial $R(x,y) = \Sigma_i=0 r_i x^i y^{s-i}$ such that \((2.1)\) is satisfied by $(P_*/R, Q_*) \cdot R$. From now on the multivariate Padé-approximant as well as the multivariate rational approximant will be called type C approximants.

3. Connection between the two approaches. First of all we remark that for the case of one variable the type-L approximant [5] as well as the type C approximant [1,2] reduce to the well-known ordinary Padé-approximant. And the polynomials $P(x,y)$ and $Q(x,y)$ satisfying \((2.1)\) do also satisfy \((1.1)\) when the sets $N, D$ and $E$ are chosen as follows:

$$N = \{(i,j) | i,j \in \mathbb{N}, lm \leq i + j \leq lm + l\},$$
$$D = \{(i,j) | i,j \in \mathbb{N}, lm \leq i + j \leq lm + m\},$$
$$E = \{(i,j) | i,j \in \mathbb{N}, lm \leq i + j \leq lm + l + m\}.$$

The set $H = E \setminus N$ has one element less than the set $D$, as required; but we could also add to $E$ the set $\{(i,j) | i,j \in \mathbb{N}, i + j < lm\}$, since $\delta_0(f \cdot Q - P) \geq lm$ for all polynomials $P$ and $Q$ as in $(2.1a)$ and $(2.1b)$. Doing so we do not impose more conditions on the coefficients $a_{ij}$ and $b_{ij}$; we write

$$E^\text{ext} = \{(i,j) | i,j \in \mathbb{N}, i + j \leq lm + l + m\}.$$

Let us now number the points in $D$ and $H$, using a diagonal enumeration:

(a) $D = \left\{ (lm,0), (lm-1,1), \ldots, (0,lm), (lm+1,0), \ldots, (0,lm+1) \right\}$, \ldots,

$$\left\{ (lm+m,0), \ldots, (0,lm+m) \right\},$$

first diagonal \hspace{1cm} second diagonal \hspace{1cm} last diagonal
When we write down the equations equivalent with condition (2.1c), the set of homogeneous equations in the unknown $b_{ij}$ has a coefficient matrix which is exactly the matrix in (1.1d) after removing the first row. From now on we will call this matrix $\mathcal{K}$; it has $p = \binom{lm+l+2}{2} - \binom{lm+2}{2}$ rows and one more column than rows.

**Theorem 3.1.** The rank of the matrix $\mathcal{K}$ is at most $p - (lm - \delta_0 Q_\ast + \min(l-l', m-m'))$.

**Proof.** We only have to prove that the dimension of the null-space of $\mathcal{K}$, which is the dimension of the space of solutions of the homogeneous system of equations, is at least $lm - \delta_0 Q_\ast + \min(l-l', m-m') + 1$; in other words that (2.1) admits solutions where at least $lm - \delta_0 Q_\ast + \min(l-l', m-m') + 1$ of the $b_{ij}$ can be freely chosen. Precisely this is formulated in Theorem 2.1.

The number $s = lm - \delta_0 Q_\ast + \min(l-l', m-m')$ is one less than the number of coefficients in a homogeneous polynomial of degree $s$ in two variables, namely $\binom{s+1}{2}$. The number of coefficients in a homogeneous polynomial of degree $s$ in $n$ variables is $\binom{s+n-1}{n-1}$. But first of all we are going to take a closer look at the matrix $\mathcal{K}$ for the type C approximants when $n=2$; in the next section we will treat the $n$-variable case with $n>2$. To examine the special structure of $\mathcal{K}$ we introduce the following notation. For $Q(x,y) = \Sigma_{i+j=lm} b_{ij} x^i y^j$ we write

$$B_{lm} = \begin{pmatrix} b_{lm,0} \\ b_{lm-1,1} \\ \vdots \\ b_{0,lm} \end{pmatrix}, \quad B_{lm+1} = \begin{pmatrix} b_{lm+1,0} \\ b_{lm,1} \\ \vdots \\ b_{0,lm+1} \end{pmatrix}, \quad \ldots, \quad B_{lm+m} = \begin{pmatrix} b_{lm+m,0} \\ b_{lm+m-1,1} \\ \vdots \\ b_{0,lm+m} \end{pmatrix}.$$  

Equations (2.1c) can now be written as

$$\begin{pmatrix} H_{l+1,lm} & H_{l,lm+1} & \cdots & H_{l+1-l,m,lm+m} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & H_{l,m,lm+m} \end{pmatrix} \begin{pmatrix} B_{lm} \\ \vdots \\ B_{lm+m} \end{pmatrix} = 0,$$

where $H_{i,j}$ is a matrix with $(i+j+1)$ rows and $(j+1)$ columns and the first column equal to the transpose of $(c_{1,0} c_{1,-1,1} \cdots c_{1,i-1} c_{0,i,0} \cdots 0)$ and the next columns equal to their previous column but with all the elements shifted down one place and a zero added on top.
To calculate the displacement rank $\alpha(\mathcal{K})$ of $\mathcal{K}$, we have to construct the lower shifted difference

$$\mathcal{K} - \mathcal{K} = \begin{pmatrix} h_{1,1} & \cdots & h_{1,p+1} \\ \vdots & & \vdots \\ h_{p,1} & \cdots & h_{p,p+1} \end{pmatrix} - \begin{pmatrix} 0 & \cdots & 0 \\ h_{1,1} & \cdots & h_{1,p} \\ \vdots & & \vdots \\ 0 & \cdots & h_{p-1,1} \end{pmatrix}$$

Now $\alpha(\mathcal{K}) = \text{rank}(\delta \mathcal{K}) + 2$ [6].

**Theorem 3.2.** The displacement-rank of the matrix $\mathcal{K}$ is at most $m + 2$.

**Proof.** Let us write down the matrix $\mathcal{K}$ more explicitly:

$$\mathcal{K} = \begin{pmatrix} c_{l+1,0} & 0 & \cdots & 0 & c_{l,0} & 0 & \cdots & 0 & c_{l+1-m,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \ddots & \cdots & \vdots \\ c_{0,l+1} & 0 & \cdots & 0 & c_{0,l} & 0 & \cdots & 0 & c_{0,l+1-m} & 0 & \cdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \ddots & \cdots & \vdots \\ 0 & \cdots & 0 & c_{0,l+1} & 0 & \cdots & 0 & c_{0,l} & 0 & \cdots & 0 & c_{0,l+1-m} \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \ddots & \cdots & \vdots \\ c_{l+m,0} & 0 & \cdots & 0 & c_{l,0} & 0 & \cdots & 0 & c_{l+1-m,0} & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \ddots & \cdots & \vdots \\ c_{0,l+m} & 0 & \cdots & 0 & c_{l+m,0} & 0 & \cdots & 0 & c_{0,l+1-m} & 0 & \cdots & \vdots \\ 0 & \cdots & 0 & c_{0,l+m} & 0 & \cdots & 0 & c_{0,l+1-m} & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \ddots & \cdots & \vdots \end{pmatrix}$$

In our case $\delta \mathcal{K}$ has the structure $\delta \mathcal{K} = (\Delta_1, \Delta_2, \ldots, \Delta_{m+1})$, where $\Delta_i$ has $(p-1)$ rows and $(lm+i)$ columns, $\Delta_i$ has $(p-1)$ rows and $(lm+i)$ columns for $i = 2, \ldots, m+1$, and only the first column in $\Delta_i$ with $i \geq 2$ may contain nonzero elements; all the other elements in $\delta \mathcal{K}$ equal zero. So $\text{rank}(\delta \mathcal{K}) \leq m$ and this proves our theorem.

We will illustrate the theorems with some very simple examples. Consider $f(x,y) = 1 + x/(0.1 - y) + \sin(xy)$.

(a) Take $l = 1 = m$. The type C approximant is

$$\frac{1 + 10x - 10 \cdot 1y}{1 - 10 \cdot 1y}$$

with $l' = 1 = m'$, $\partial_0 \mathcal{Q}_s = 0$, $s = 1$ and $\alpha(\mathcal{C}) = 3$. 
The matrix
\[
\mathcal{C} = \begin{bmatrix}
0 & 0 & 10 & 0 & 0 \\
101 & 0 & 0 & 10 & 0 \\
0 & 101 & 0 & 0 & 10 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix};
\]
its rank is 3.

(b) Take \(l=4\) and \(m=2\). The type C approximant is
\[
\frac{1 + 10x - 10y + xy - 10xy^2}{1 - 10y}
\]
with \(l'=3\), \(m'=1\), \(\delta_0 Q_s = 0\), \(s = 9\) and \(\alpha(\mathcal{C}) = 4\).

The matrix
\[
\mathcal{C} = \begin{bmatrix}
H_1 & H_2 & H_3 \\
H_4 & H_5 & H_6
\end{bmatrix},
\]
where
\[
H_1 = 10^5 [\delta_{i,j+4}], \text{ a } 14 \times 9 \text{ matrix},
H_2 = 10^4 [\delta_{i,j+3}], \text{ a } 14 \times 10 \text{ matrix},
H_3 = 10^3 [\delta_{i,j+2}], \text{ a } 14 \times 11 \text{ matrix},
H_4 = 10^6 [\delta_{i,j+5}] - \frac{1}{6} [\delta_{i,j+3}], \text{ a } 15 \times 9 \text{ matrix},
H_5 = 10^5 [\delta_{i,j+4}], \text{ a } 15 \times 10 \text{ matrix},
H_6 = 10^4 [\delta_{i,j+3}], \text{ a } 15 \times 11 \text{ matrix},
\]
and \(\delta_{i,j}\) is the Kronecker symbol (here used in rectangular matrices). \(\mathcal{C}\) is a matrix of rank 20.

(c) Take \(l=1\) and \(m=2\). The type C approximant is
\[
\frac{x - 1.01y + 10y^2 + 10x^2 - 20.2xy}{x - 1.01y + 10y^2 - 10.1xy + 2.01xy^2}
\]
with \(l'=1\), \(m'=2\), \(\delta_0 Q_s = 1\), \(s = 1\) and \(\alpha(\mathcal{C}) = 4\). The matrix
\[
\mathcal{C} = \begin{bmatrix}
0 & 0 & 0 & 10I_4 & I_5 \\
101I_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1000I_3 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
where \(I_k\) is the \(k \times k\) unit-matrix. The rank of \(\mathcal{C}\) is 10.

(d) Consider \(f(x,y) = (xe^x - ye^y)/x - y\) and take \(l=1=m\). The determinant representations yield
\[
P(x,y) = -\frac{1}{2}(x+y+0.5x^2 + 1.5xy + 0.5y^2),
Q(x,y) = -\frac{1}{2}(x+y-0.5x^2 - 0.5xy - 0.5y^2),
\]
and indeed the type C approximant is

\[ \frac{P(x, y)}{Q(x, y)} = \frac{P_*(x, y)}{Q_*(x, y)} \]

with \( \partial_0 Q_* = 1 \) and \( l' = 1 = m' \). The matrix \( \mathcal{C} \) has rank \( p \) (\( s = 0 \)) and \( \alpha(\mathcal{C}) = 3 \).

4. The multivariate case. Given the power series

\[ f(x) = \sum_{k=0}^{\infty} c_k x^k, \]

where \( x = (x_1, \cdots, x_n) \in \mathbb{R}^n \), \( c_k = c_{k_1} \cdots c_{k_n} \) \( x^k = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} \) and \( \Sigma_{k=0}^{\infty} = \Sigma_{k_1=0}^{\infty} \cdots \Sigma_{k_n=0}^{\infty} \), the Padé-approximation problem of order \((l, m)\) is the following:

find

\[
\begin{align*}
(4.1a) \quad P(x) &= \sum_{|i| = lm} a_i x^i, \\
(4.1b) \quad Q(x) &= \sum_{|i| = lm} b_j x^i
\end{align*}
\]

where \( a_i = a_{i_1, \cdots, i_n} \) and \( b_j = b_{j_1, \cdots, j_n} \), \(|i| = i_1 + \cdots + i_n \) and \(|j| = j_1 + \cdots + j_n \), such that

\[ \partial_0 (f \cdot Q - P) = lm + l + m + 1 \]

where \( \partial_0 \) is again the degree of the first nonzero term.

After calculation of the nontrivial solution of (4.1) [2] we can proceed as in §2 and define the multivariate Padé-approximant of order \((l, m)\) and the multivariate rational approximant of order \((l, m)\).

The integers \( l' \) and \( m' \) are defined as in the two-variable case and it is easy to prove the following \( n \)-dimensional analogue of Theorem 2.1.

**Theorem 4.1.** For the irreducible form \( P_*/Q_* \) of \( P/Q \) where \( (P, Q) \) satisfies (4.1) and for every polynomial \( R(x) = \sum_{|i| = r} x^i \) with \( s = lm - \partial_0 Q_* + \min(l - l', m - m') \), \( (P_*, R, Q_*, R) \) satisfies (4.1).

Let us again study the connection with the approach of Levin. Condition (4.1c) results in \( (\frac{n+lm+l+m}{lm+lm+m}) \) equations: the first \( (\frac{n+lm+l+m}{lm+lm+m}) \) equations express the \( a_i \) as linear combinations of the \( b_j \) and the remaining equations form an overdetermined homogeneous linear system in the unknown \( b_j \) [2]; there are

\[ p + 1 = \left( \frac{n+lm+m}{lm+m} \right) - \left( \frac{n+lm-1}{lm-1} \right) \]

unknown coefficients \( b_j \). The \( b_j \) can be found by solving a homogeneous subsystem of \( p \) equations, having the rank of the overdetermined system.

Choose the sets \( N, D \) and \( H \) as follows:

- \( N = \{i = (i_1, \cdots, i_n) | i \in \mathbb{N}^n, lm \leq |i| \leq lm + l\} \);
- \( D = \{i = (i_1, \cdots, i_n) | i \in \mathbb{N}^n, lm \leq |i| \leq lm + m\} \);
- select a particular \( b_j \) and let \( c_{h(k)-j} \) be the coefficient of \( b_j \) in the \( k \)th equation of the homogeneous subsystem we have to solve \((k = 1, \cdots, p)\),

\[ H = \{h(k) = (h_1(k), \cdots, h_n(k)) | k = 1, \cdots, p\}. \]

We call the coefficient matrix of the homogeneous subsystem again \( \mathcal{C} \). It is easy to prove the following \( n \)-dimensional analogue of Theorem 3.1.
Theorem 4.2. The rank of the matrix $\mathcal{K}$ is at most $p-(s+n-1)+1$ with $s=lm-\delta_0 Q_\ast + \min(l-l',m-m')$.

If we use an enumeration of the points in $D$ and $H$, similar to the one described in §3, it is obvious that in the multivariate case $\mathcal{K}$ is also a matrix with low displacement rank.

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REFERENCES


