A PROJECTION-PROPERTY FOR ABSTRACT RATIONAL (1-POINT) APPROXIMANTS

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1. NOTATION AND DEFINITIONS

Consider the operator \( F: X \to Y \), analytic in \( 0 \) [2, pp. 113] where \( X \supseteq \{0\} \) is a Banach space and \( Y \supseteq \{0, I\} \) is a commutative Banach algebra without nilpotent elements (0 is the unit for addition and \( I \) is the unit for multiplication). The scalar field is \( \mathbb{R} \) or \( \mathbb{C} \).

A nonlinear operator \( P: X \to Y \) such that \( P(x) = A_n x^n + \ldots + A_0 \) with \( A_i: X^i \to Y \) a symmetric and bounded \( i \)-linear operator \( (i = 0, \ldots, n) \) is called an abstract polynomial [2, pp. 111]. The degree of \( P(x) \) is \( n \). The notation for the exact degree of \( P(x) \) is \( \partial P \) (the largest integer \( k \) with \( A_k x^k \neq 0 \)) and the notation for the order of \( P(x) \) is \( \delta P \) (the smallest integer \( k \) with \( A_k x^k \neq 0 \)).

Write \( D(F) := \{ x \in X | F(x) \text{ is regular in } Y, \text{i.e. there exists } y \in Y: F(x) \cdot y = I \} \). Since \( F \) is analytic in \( 0 \), there exists \( r > 0 \) such that

\[
F(x) = \sum_{k=0}^{\infty} \frac{1}{k!} F^{(k)}(0) x^k \quad \text{for } \|x\| < r.
\]

We say that \( F(x) = O(x^j) \) for \( j \in \mathbb{N} \) if there exist \( J \in \mathbb{R}^+_0 \) and \( 0 < r < 1 \) such that

\[
\|F(x)\| \leq J \|x\|^j.
\]

Definition 1.1. The couple of abstract polynomials

\[
(P(x), Q(x)) := (A_{nm+n}x^{nm+n} + \ldots + A_{mn}x^{mn}, B_{nm+n}x^{nm+m} + \ldots + B_{nm}x^{nm})
\]

is called a solution of the Padé approximation problem of order \((n, m)\) for \( F \) if the abstract power series

\[
(F \cdot Q - P)(x) =: O(x^{nm+n+m+1}).
\]
We define the operator \( \frac{1}{Q} \cdot D(Q) \rightarrow Y \) by \( \frac{1}{Q}(x) := [Q(x)]^{-1} \cdot \) the inverse element of \( Q(x) \) for the multiplication in \( Y \). We call the abstract rational operator \( \frac{1}{Q} \cdot P \), the quotient of two abstract polynomials, reducible if there exist abstract polynomials \( T, R \) and \( S \) such that \( P \equiv T \cdot R, \ Q \equiv T \cdot S \) and \( cT \ni 1 \).

Let us assume that the Banach space \( X \) and the Banach algebra \( Y \) are such that the irreducible form of an abstract rational operator is unique and that the abstract rational approximant of order \((n, m)\) for \( F \) (see Definition 1.2) is unique. The matter was discussed in [1].

**Definition 1.2.** Let \((P, Q)\) be a couple of abstract polynomials satisfying Definition 1.1, with \( D(P) \cup D(Q) \neq \emptyset \). The irreducible form \( \frac{1}{Q*} \cdot P* \) of \( \frac{1}{Q} \cdot P \) is called the \textit{abstract rational approximant of order \((n, m)\)} for \( F \) (abbreviated \((n, m)\)-ARA).

To prove our projection-property we shall need the condition numbered (1). Let \( T(x) := \sum_{k=0}^{n} T_k x^k \) be the abstract polynomial such that \( P := P* \cdot T \) and \( Q := Q* \cdot T \). Because \( D(P) \cup D(Q) \neq \emptyset \) we have \( D(T) \neq \emptyset \). If

\[
(1) \quad D(T_{\omega^t}) \neq \emptyset
\]

then we have \( t \geq 0 \) such that

\[
(F \cdot Q* \cdot P)_{\omega t} := O(x^{\tilde{c}^t P* + \tilde{c}^t Q* + \tilde{c}^t \tilde{c}^t + \cdots + \tilde{c}^t})
\]

\[
\tilde{c}^t P* \leq n \leq \tilde{c}^t P_{\omega t} + t
\]

\[
\tilde{c}^t Q* \leq m \leq \tilde{c}^t Q_{\omega t} + t
\]

where \( \tilde{c}^t P_{\omega t} := \tilde{c}^t P_{\omega t} \cdots \tilde{c}^t \tilde{c}^t \) and \( \tilde{c}^t Q_{\omega t} := \tilde{c}^t Q_{\omega t} \cdots \tilde{c}^t \tilde{c}^t \) [1, pp. 208].

2. **Projection-Property**

Consider Banach spaces \( X_i \) \((i = 1, \ldots, p)\). The space \( \prod_{i=1}^{p} X_i \) normed by one of the following Minkowski norms

\[
\|x^{[q]}_{i, q}^t := \left( \sum_{i=1}^{p} \|x^{[q]}_{i, q}^t \right)^{1/q}
\]
or
\[ \|x\|_1 = \sum_{i=1}^{p} \|x_i\|_{(i)} \]
or
\[ \|x\|_{\infty} = \max(\|x_1\|_{(1)}, \ldots, \|x_p\|_{(p)}) \]
where \( \|x_i\|_{(i)} \) is the norm of \( x_i \) in \( X_i \) and \( x = (x_1, \ldots, x_p) \), is also a Banach space.

We introduce the notations
\[ x_{(j)} = (x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_p) \]
\[ \hat{x}_{(j)} = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_p) . \]

**Theorem 2.1.** Let \( X = \prod_{i=1}^{p} X_i \) and \( \left( \frac{1}{Q_*} \cdot P_* \right)(x) \) be the \((n,m)\)-ARA for \( F \) and \( j \in \{1, \ldots, p\} \).

Let (1) be satisfied. If
\[ S(\hat{x}_{(j)}) := Q_*(x_{(j)}) \]
\[ R(\hat{x}_{(j)}) := P_*(x_{(j)}) \]
\[ D(S) \cup D(R) \neq \emptyset \]
\[ G_j(\hat{x}_{(j)}) := F(x_{(j)}) \]
then the irreducible form \( \left( \frac{1}{S_*} \cdot R_* \right)(\hat{x}_{(j)}) \) of \( \left( \frac{1}{S} \cdot R \right)(\hat{x}_{(j)}) \) is the \((n,m)\)-ARA for \( G_j \).

**Proof.** First we remark that if \( L : X^k \to Y \) is a bounded \( k \)-linear operator, then the operator \( M : \left( \prod_{i=1}^{p} X_i \right)^k \to Y \) defined by \( M\hat{x}_{(j)}^k = Lx_{(j)}^k \) is also bounded and \( k \)-linear.

Since \( \left( \frac{1}{Q_*} \cdot P_* \right)(x) \) is the \((n,m)\)-ARA for \( F \) and since (1) is satisfied, we have \( \epsilon > 0 \) such that
\[ (F \cdot Q_* - P_*) (x) = O(x_{\epsilon_1}^{\epsilon_1 P_* + \epsilon_1 Q_* + \epsilon_0 Q_* + \epsilon_1 + 1}) \]
\[ \partial_1 P_* < n \leq \partial_1 P_* + t \]
\[ \partial_1 Q_* < m \leq \partial_1 Q_* + t . \]
Using one of the Minkowski norms \( \| \cdot \|_q \) (1 \( \leq q \leq \infty \)), \( \| x_{(j)} \|_q = \| (x_j, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_p) \|_q \) in \( \prod_{i=1}^{p} X_i \) equals \( \| \hat{x}_{(j)} \|_q = \| (x_j, \ldots, x_{j-1}, x_{j+1}, \ldots, x_p) \|_q \).

Thus

\[
(F \cdot Q_\ast - P_\ast)(x_{(j)}) = (G_j \cdot S - R)(\hat{x}_{(j)}) = \mathcal{O}(\hat{x}_{(j)}^{(\partial P_\ast + \partial Q_\ast \cdot \partial Q_\ast \cdot t + t)})
\]

Now \( \partial P_\ast = \partial_0 Q_\ast + \partial_1 P_\ast \leq \partial P - \partial_0 T \leq nm + n \) [1, pp. 199], and \( \partial Q_\ast = \partial_0 Q_\ast + \partial_1 Q_\ast \leq \partial Q - \partial_0 T \leq nm + m \) [1, pp. 199]. So \( s = nm - \partial_0 Q_\ast + \min(n - \partial_1 P_\ast, m - \partial_1 Q_\ast) \geq 0 \).

Take a bounded \( s \)-linear operator \( D_s : \left( \prod_{i=1}^{p} X_i \right)^{s} \rightarrow Y \) with \( D(D_s) \cap [D(S) \cup D(R)] \neq \emptyset \).

Then

\[
\partial_0 (S \cdot D_s) \geq nm
\]

\[
\partial_0 (R \cdot D_s) \geq nm
\]

\[
\partial (S \cdot D_s) \leq \partial_0 Q_\ast + \partial_1 Q_\ast + nm - \partial_0 Q_\ast + \min(n - \partial_1 P_\ast, m - \partial_1 Q_\ast) \leq nm + m
\]

\[
\partial (R \cdot D_s) \leq \partial_0 Q_\ast + \partial_1 P_\ast + nm - \partial_0 Q_\ast + \min(n - \partial_1 P_\ast, m - \partial_1 Q_\ast) \leq nm + n
\]

\[
[(G_j \cdot S - R) \cdot D_s](\hat{x}_{(j)}) = \mathcal{O}(\hat{x}_{(j)}^{(\partial P_\ast + \partial_1 Q_\ast \cdot \partial Q_\ast \cdot nm + t \cdot \min(n - \partial_1 P_\ast, m - \partial_1 Q_\ast + 1)}) \leq \mathcal{O}(\hat{x}_{(j)}^{(nm + n + m + 1)})
\]

since \( m \leq \partial_1 Q_\ast \cdot t \) and \( n \leq \partial_1 P_\ast \cdot t \). The irreducible form of \( \frac{1}{S \cdot D_s} \) is the irreducible form of \( \frac{1}{S} \cdot R \).

We give a simple example to illustrate the theorem. Take

\[
G: \mathbb{R}^2 \rightarrow \mathbb{R}: (x, y) \rightarrow \frac{x \exp(x) \cdots y \exp(y)}{x \cdots y}
\]
The \((1,1)\)-ARA for \(G\) is
\[
\frac{x + y + 0.5(x^2 + 3xy + y^2)}{x + y - 0.5(x^2 + xy + y^2)}.
\]

For \(j = 1\): \(x = 0\)
\[G_1: \mathbb{R} \to \mathbb{R}: y \to \exp(y)\]
and for \(j = 2\): \(y = 0\)
\[G_2: \mathbb{R} \to \mathbb{R}: x \to \exp(x)\]

Indeed the \((1,1)\)-ARA for \(G_1\) equals \(\frac{1 + 0.5y}{1 - 0.5y}\) and for \(G_2\) equals \(\frac{1 + 0.5x}{1 - 0.5x}\).

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REFERENCES


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