Geometric functional analysis

Academic year 2017-18
(autumn term)

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Motivation and introduction

1. What is geometric functional analysis?

One of the fundamental tasks of mathematics is the study of maps from one space to another space. One dimensional calculus focuses on functions

\[ f : \mathbb{R} \to \mathbb{R} \]

and multivariable calculus on functions

\[ f = (f_1, \ldots, f_n) : \mathbb{R}^m \to \mathbb{R}^n. \]

Functional analysis studies maps between more general spaces, usually infinite dimensional vector spaces with Banach or Hilbert structure. Such maps are often referred to as ‘operators’ or, if the target space is \( \mathbb{R} \) or \( \mathbb{C} \), as ‘functionals’. If one considers maps between infinite dimensional Banach and Hilbert manifolds instead of Banach and Hilbert vector spaces one often speaks of global analysis rather than functions analysis. Geometric functional analysis aims at presenting the theorems and methods of ‘classical’ functional analysis and showing their applications in geometry and topology.

We will try now to get an impression of examples, applications, and strength of geometric functional analysis in topology and dynamical systems. Since each of the two following examples is in fact an important theory on its own, we will only give a rough outline of the constructions and phenomena. Later in the course, we will sometimes come back and study specific aspects in detail.

2. Applications in topology: Morse theory

2.1. Motivation. For standard notions from differential geometry like (Riemannian) manifold, (Riemannian) metric, and Hessian etc., we refer to Definition A.7 in the appendix.

Definition 1.1. Let \( N \) be a smooth manifold and \( f \in C^2(N, \mathbb{R}) \). A point \( x \in N \) is critical for \( f \) if the derivative vanishes in \( x \), i.e., \( Df|_x = 0 \). We denote the set of critical points by

\[ \text{Crit}(f) := \{ x \in N \mid Df|_x = 0 \} \]
and call $x \in \text{Crit}(f)$ nondegenerate if the Hessian $D^2 f\big|_x$ is nondegenerate. $f$ is a Morse function if all points in $\text{Crit}(f)$ are nondegenerate.

Let us consider $S^2 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ and its ‘height function’ $f : S^2 \to \mathbb{R}$ given by $f(x, y, z) := z$. If we study the (change of) level sets $f^{-1}(r)$ for $r \in \mathbb{R}$ we observe

$$f^{-1}(r) = \begin{cases} 
\emptyset, & r \in ]-\infty, -1[, \\
1 \text{ point}, & r = -1, \\
1 \text{ circle}, & r \in ]-1, 1[, \\
1 \text{ point}, & r = 1, \\
\emptyset, & r \in ]1, \infty[, 
\end{cases}$$

i.e., the level sets undergo a drastic change precisely at the two critical points of $f$ given by the ‘south pole’ $(0, 0, -1)$ and the ‘north pole’ $(0, 0, 1)$. If we consider now the height function of a deformed sphere, we get even more possible shapes of level sets and transitions between them — for example ‘figure eight’ shaped curves when switching from one circle to two circles and vice versa. But depending on the positioning of the deformed sphere in space we may also not notice the deformation at all by observing the change of level sets!

Studying other examples makes us nevertheless notice that level sets (usually) change at levels that contain critical points. This motivates the question:

**Question 1.2.** Can we somehow analyse the topology of a manifold $N$ by studying the critical points of a smooth function $f : N \to \mathbb{R}$? Or, vice versa, does the topology of the manifold enforce constraints on the number and positioning of the critical points of $f : N \to \mathbb{R}$?

The general answer is yes and the specific answer what is influenced by what is described by Morse theory. It is beyond the scope of this course to present Morse theory in detail for which we refer to the literature [Audin & Damian], [Milnor], and [Schwarz], but we want to give at least a rough sketch.
CHAPTER 2

Classical theorems of functional analysis

1. Baire’s theorem and genericity

In this section, we will study the following question: Given a topological space $X$ (cf. Definition A.1) and a subset $V \subseteq X$, we would like to measure ‘how much’ of the space $X$ is actually contained in $V$. In particular, we are interested in subsets $V$ that ‘fill almost everything’ of the space $X$, i.e., we are looking for a notion of ‘large’ and ‘small’ in the following sense:

- $V \subseteq X$ is ‘large’ $\iff X \setminus V$ is ‘small’.
- Countable unions of large sets are large.
- $V \subseteq X$ ‘large’ $\Rightarrow V \neq \emptyset$.

If $X$ carries the structure of a measure space with finite measure, i.e., if we have a triple $(X, \mathcal{A}, \mu)$ with $\sigma$-algebra $\mathcal{A}$ and finite measure $\mu$, the natural notion of ‘large’ for $V \in \mathcal{A}$ would be $\mu(A) = \mu(X)$ and for ‘small’ $\mu(V) = 0$.

Is there also a meaningful notion if the space is not endowed with a measure? The rough answer is yes and we now explain under which conditions.

First we recall without proof the following notion from general set topology.

**Definition and Lemma 2.1.** Let $X$ be a topological space. $V \subseteq X$ is **dense in** $X$ if one of the following equivalent conditions is satisfied.

1. $\overline{V} = X$.
2. $V \cap O \neq \emptyset$ for all open, nonempty sets $O \subseteq X$.

We are now interested in the following type of sets.

**Definition and Lemma 2.2.** Let $X$ be a topological space. $V \subseteq X$ is **nowhere dense in** $X$ if one of the following equivalent conditions is satisfied.

1. $\text{Int}(\overline{V}) = \emptyset$.
2. $X \setminus V$ contains an open dense subset.
3. For all open, nonempty $G \subseteq X$, there exists an open, nonempty $\tilde{G} \subseteq G$ such that $\tilde{G} \cap V = \emptyset$.

**Proof.** $(1) \Rightarrow (2)$: The set $U := X \setminus \overline{V} \subseteq X \setminus V$ is open and dense by construction.
2. CLASSICAL THEOREMS OF FUNCTIONAL ANALYSIS

(2) ⇒ (3): Let \( U \subseteq X \setminus V \) be open and dense and let \( G \subseteq X \) be open and nonempty. Then the subset \( \tilde{G} := G \cap U \) is open and \( \tilde{G} \cap V = \emptyset \) by construction.

(3) ⇒ (1): Assume that \( \text{Int}(\overline{V}) \neq \emptyset \). By definition, \( \text{Int}(\overline{V}) \) is open and we find for all open \( \tilde{G} \subseteq \text{Int}(\overline{V}) \) that \( \tilde{G} \cap V \neq \emptyset \). \( \square \)

Obviously \( V \subseteq X \) is nowhere dense if and only if \( \overline{V} \) is nowhere dense. Now we come to the main definitions of this paragraph.

**Definition 2.3.** Let \( X \) be a topological space and \( V \subseteq X \).

1) \( V \) is **meager** or of **first (Baire) category** if \( V \) is a countable union of nowhere dense sets.

2) \( V \) is **nonmeager** or of **second (Baire) category** if \( V \) is not of first (Baire) category.

This property is named after the French mathematician René-Louis Baire (1871 – 1932). Intuitively, meager sets are ‘small’ and nonmeager sets ‘large’.

**Remark 2.4.** Let \( X \) be a topological space.

1) If \( V \subseteq X \) is meager and \( \tilde{V} \subseteq V \) then \( \tilde{V} \) is also meager.

2) If, for all \( n \in \mathbb{N} \), the sets \( V_n \subseteq X \) are meager and \( \tilde{V}_n \subseteq V_n \) then \( \bigcup_{n \in \mathbb{N}} \tilde{V}_n \) is also meager.

In geometric applications, one is usually more interested in nonmeager sets than in the meager ones. This prompts the following definition.

**Definition 2.5.** A topological space \( X \) is said to be **Baire** or a **Baire space** if the countable intersection of open and dense subsets of \( X \) is always dense.

There are in fact several equivalent ways to define Baire spaces.

**Theorem 2.6.** Let \( X \) be a topological space. Then the following statements are equivalent.

1) \( X \) is Baire.

2) The countable union of closed, nowhere dense subsets of \( X \) has empty interior.

3) Every meager subset of \( X \) has empty interior.

4) The complement of every meager subset of \( X \) is dense in \( X \).

5) Every nonempty open subset of \( X \) is not meager in \( X \).

**Proof.** We leave this proof as an exercise to the reader since it is of purely topological nature and does not lie within the scope of this course. The equivalence of items (1) and (2) can be found for example in [Chacón & Rafeiro & Vallejo, Th. 9.3]. \( \square \)

Now we look for ‘natural’ Baire spaces.
**Theorem 2.7 (Baire).** 1) Every complete metric space is Baire.
2) Every locally compact Hausdorff space is Baire.

**Proof.** 1) Let \((X, d)\) be a complete metric space and \(V_n \subseteq X\) open and dense for all \(n \in \mathbb{N}\). We have to show that \(V := \bigcap_{n \in \mathbb{N}} V_n\) is dense in \(X\), i.e., for any open subset \(B \subseteq X\), we have \(B \cap V \neq \emptyset\).

Since \(B\) is open and \(V_1\) is open and dense there exists a point \(x_1 \in B \cap V_1\). Since \(B \cap V_1\) is open there exists \(0 < \varepsilon_1 < 1\) such that \(\overline{B}(x_1, \varepsilon_1) \subseteq B \cap V_1\). Since \(B_1\) is open and \(V_2\) is open and dense there exists \(x_2 \in B_1 \cap V_2\) and \(0 < \varepsilon_2 < \frac{1}{2}\) such that \(\overline{B}(x_2, \varepsilon_2) \subseteq B_1 \cap V_2\). Iterating this procedure, we get a nested sequence

\[
B \supset \overline{B}_1 \supset B_1 \supset \overline{B}_2 \supset B_2 \supset \cdots \supset B_n \supset \overline{B}_{n+1} \supset B_{n+1} \supset \ldots
\]

whose radii satisfy \(0 < \varepsilon_n < \frac{1}{n}\) and therefore converge to zero. The sequence of middle points \((x_n)_{n \in \mathbb{N}}\) is a Cauchy sequence since the nestedness implies \(d(x_k, x_l) \leq \frac{2}{N}\) for all \(k, l \geq N\). Since \((X, d)\) is complete \(x := \lim_{n \to \infty} x_n\) exists and lies by construction in \(\bigcap_{n \in \mathbb{N}} \overline{B}_n\) which in turn lies in \(B \cap V\).

2) Left as an exercise to the reader. A proof may be found in [Lowen]. \(\square\)

**Example 2.8.** Banach spaces are Baire.

**Proof.** The complete norm \(|\|\|\) of the Banach space gives rise to a complete metric \(d\) via \(d(x, y) := ||x - y||\). \(\square\)

In applications, often the following notion is used.

**Definition 2.9.** Let \(X\) be a topological space. \(V \subseteq X\) is called **generic** if \(V\) is dense in \(X\) and can be written as the countable intersection of open sets.

Being ‘generic’ means intuitively to show ‘typical behaviour’ or to ‘behave as most functions in a given situation behave’.

Now we want to know what kind of applications Theorem 2.7 (Baire) and the notion ‘generic’ have in analysis and geometry. First we recall the following phenomenon.

**Remark 2.10.** There exist continuous functions that are nowhere differentiable, for example

- The Weierstraß function \(W : \mathbb{R} \to \mathbb{R}\) given by

\[
W(x) := \sum_{n=0}^{\infty} b^n \cos(a^n \pi x)
\]

with \(a \in \mathbb{N}\) odd and \(b \in ]0, 1[\) satisfying \(ab > 1\) and \(\frac{2}{3} > \frac{n}{ab-1}\).
• The Riemann type function
  \[ x \mapsto \sum_{n=0}^{\infty} \frac{1}{n^2} \sin(n^2 \pi x). \]

• Van der Waerden’s function
  \[ x \mapsto \sum_{n=0}^{\infty} \inf_{k \in \mathbb{Z}} \frac{|4^n x - k|}{4^n}. \]

Now we want to answer the immediate question if being nowhere differentiable is a ‘typical’ or ‘rare’ property of continuous functions, i.e., is it generic or not?

**Theorem 2.11.**

1) The set \( \{ f \in C^0([0, 1], \mathbb{R}) \mid f \text{ nowhere differentiable} \} \) is generic in \( C^0([0, 1], \mathbb{R}) \).

2) The set \( \{ f \in C^0([0, 1], \mathbb{R}) \mid f \text{ somewhere differentiable} \} \) is meager in \( C^0([0, 1], \mathbb{R}) \).

**Proof.**

1) Abbreviate \( I := [0, 1] \) and define for \( g \in C^0(I, \mathbb{R}) \) and \( x, y \in I \) satisfying \( x + y \in I \) the function
  \[ F_n^g(x, y) := |g(x + y) - g(x)| - n|y| \]

which measures how much the difference quotient in the definition of differentiability blows up. Moreover, define the subsets
  \[ U_n := \{ g \in C^0(I, \mathbb{R}) \mid \forall x \in I \exists y \in I : F_n^g(x, y) > 0 \} \]

whose intersection \( U := \bigcap_{n \in \mathbb{N}} U_n \) consists of functions that are nowhere in \( I \) differentiable. We have to show that \( U \) is generic.

**Step 1:** \( C^0(I, \mathbb{R}) \) equipped with \( d(f, g) := \|f - g\|_\infty := \sup_{x \in I} |f(x) - g(x)| \) is a complete metric space. Using Theorem 2.7 (Baire), it is enough to show that all \( U_n \) are open and dense in \( C^0(I, \mathbb{R}) \).

**Step 2:** We show that \( U_n \) is open, i.e., given \( g \in U_n \), we show that there exists a small open ball around \( g \) that lies in \( U_n \).

We notice that, for each \( g \in U_n \), there exists \( \varepsilon > 0 \) such that for all \( x \in I \)
there is \( y = y(x) \in I \) with \( F_n^g(x, y) > \varepsilon \). Note that this \( \varepsilon \) holds uniformly for all \( x \in I \). The inequality \( F_n^g(x, y) > \varepsilon \) can be reformulated as
  \[ \varepsilon + n|y| < |g(x + y) - g(x)| \]

which gives rise to the following estimate for all \( h \in C^0(I, \mathbb{R}) \):

\[ \varepsilon + n|y| < |g(x + y) - g(x)| \leq |g(x + y) - h(x + y)| + |h(x + y) - h(x)| + |h(x) - g(x)| \leq \|g - h\|_\infty + F_n^h(x, y) + n|y| + \|g - h\|_\infty \]
This implies the inequality \(0 < F^h_n(x, y)\) for all \(h\) with \(\|g - h\|_\infty < \frac{\varepsilon}{2}\). Thus the whole open ball around \(g\) with radius \(\frac{\varepsilon}{2}\) lies in \(U_n\), i.e., \(U_n\) is open.

**Step 3:** We have to show that \(U_n\) is dense in \(C^0(I, \mathbb{R})\), i.e., given \(g \in C^0(I, \mathbb{R})\) and \(\varepsilon > 0\), there exists \(h \in U_n\) such that \(\|g - h\|_\infty < \varepsilon\).

It is easy to see that the set of piecewise linear continuous functions is dense in \(C^0(I, \mathbb{R})\). Therefore it is enough to work with a piecewise linear \(g\) from now on. Since \(I\) is compact there exists \(K > 0\) such that \(g' \leq K\) wherever the derivative \(g'\) is defined. Consider the continuous function

\[
\varphi : I \to \mathbb{R}, \quad \varphi(x) := \inf_{k \in \mathbb{Z}} |x - k|
\]

and choose \(m \in \mathbb{N}\) such that \(\varepsilon m - K > n\). The function

\[
h : I \to \mathbb{R}, \quad h(x) := g(x) + \varepsilon \varphi(mx)
\]

is continuous and, for \(0 \leq x < 1\), we find

\[
\lim_{y \to 0^+} \left| \frac{h(x + y) - h(x)}{y} \right| = \lim_{y \to 0^+} \left| \frac{g(x + y) + \varepsilon \varphi(m(x + y)) - g(x) - \varepsilon \varphi(mx)}{y} \right|
\]

\[
= \lim_{y \to 0^+} \left| \frac{g(x + y) - g(x)}{y} + \varepsilon \varphi(mx + my) - \varphi(mx) \right|
\]

\[
\geq |K - \varepsilon m| > n
\]

which implies \(h \in U_n\). Moreover, we estimate

\[
\|g - h\|_\infty = \sup_{x \in I} \varepsilon \varphi(mx) \leq \frac{\varepsilon}{2} < \varepsilon.
\]

2) follows immediately from 1). \(\square\)

For the construction of Morse theory, one makes use of Morse functions. Thus the question arises ‘how many’ Morse functions there are on a compact manifold.

**Theorem 2.12.** Let \(N\) be a smooth compact manifold without boundary. Then the set of Morse functions is generic in \(C^\infty(N, \mathbb{R})\).

**Proof.** This will be done either during the exercise sessions or homework or the interested reader may browse the standard literature on Morse theory like [Audin & Damian], [Milnor], [Schwarz]... \(\square\)

**Remark 2.13.** Intuitively, properties are generic if they persist under small perturbations. For example, properties like

- expression \(\neq 0\) among continuous functions,
- transversal intersections of smooth graphs

stay true after small perturbations and are thus (usually) generic. Conversely
2. Uniform boundedness and the Banach-Steinhaus theorem

In this section, we study an extension to Banach spaces of the following well-known result:

Let \( f : K \subset \mathbb{R} \to \mathbb{R} \) be continuous. Then, if \( K \) is compact, \( f \) is **uniformly continuous**, i.e., for all \( \varepsilon > 0 \) there is \( \delta = \delta(\varepsilon) > 0 \) such that for all \( x, y \in K \) with \( |x - y| < \delta \) follows \( |f(x) - f(y)| < \varepsilon \).

Let us introduce some notations before we investigate under which conditions ‘local, pointwise behaviour’ may lead to ‘controlled global behaviour’.

**Definition 2.14.** Let \((X, \|\cdot\|_X)\) and \((Y, \|\cdot\|_Y)\) be normed vector spaces over a field \( \mathbb{F} \) and \( T : X \to Y \) a map.

1) \( T \) is a **linear operator** if
\[
T(\lambda x + \tilde{x}) = \lambda T(x) + T(\tilde{x}) \quad \forall \; x, \tilde{x} \in X, \; \forall \; \lambda \in \mathbb{F}.
\]

2) A linear operator \( T \) is **bounded** if
\[
\exists \; C > 0 : \quad \|T(x)\|_Y \leq C \|x\|_X \quad \forall \; x \in X.
\]

3) The set of linear bounded operators from \( X \) to \( Y \) is denoted by \( \mathcal{B}(X, Y) \).

4) The **operator norm** of \( T \in \mathcal{B}(X, Y) \) is given by
\[
\|T\| := \sup\{\|T(x)\|_Y \mid x \in X, \|x\|_X \leq 1\}.
\]

5) The space \( \mathcal{B}(X, \mathbb{R}) \) and \( \mathcal{B}(X, \mathbb{C}) \) are the real and complex **dual space** of \( X \). Sometimes they are denoted by \( X^* \) or \( X' \) in the literature.

We recall the following standard facts.

**Remark 2.15.** 1) For a linear operator, being bounded is equivalent to being continuous which in turn is equivalent to being continuous at the origin, cf. for instance [Hohloch1].

2) An equivalent definition of the operator norm is
\[
\|T\| = \sup\{\|T(x)\|_Y \mid x \in X, \|x\|_X = 1\}.
\]

3) The operator norm depends on the choice of the norms \( \|\cdot\|_X \) and \( \|\cdot\|_Y \).

4) If \( Y \) is a Banach space then \( \mathcal{B}(X, Y) \) is Banach, too. In particular, the dual spaces \( \mathcal{B}(X, \mathbb{R}) \) and \( \mathcal{B}(X, \mathbb{C}) \) are Banach.
Now we want to study boundedness for families of linear operators. Given an index set $J$, we abbreviate a family

$$T_j \in \mathcal{B}(X, Y) \text{ for all } j \in J$$

by

$$(T_j)_{j \in J} \in \mathcal{B}(X, Y).$$

**Definition 2.16.** Let $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ be normed vector spaces, $J$ an index set, and $(T_j)_{j \in J} \in \mathcal{B}(X, Y)$.

1) The family $(T_j)_{j \in J}$ is **pointwise bounded** if

$$\forall x \in X \exists C_x > 0 : \|T_j(x)\|_Y \leq C_x \quad \forall j \in J.$$

2) The family $(T_j)_{j \in J}$ is **uniformly bounded** if

$$\exists C > 0 : \|T_j(x)\|_Y \leq C \quad \forall j \in J.$$

The following crucial theorem is named after Stefan Banach (Polish mathematician, 1892 – 1945) and Hugo Dyonizy Steinhaus (Polish mathematician, 1887 – 1972).

**Theorem 2.17 (Banach-Steinhaus).** Let $(X, \| \cdot \|_X)$ be a Banach space and $(Y, \| \cdot \|_Y)$ a normed vector space. Let $J$ be an index set and $(T_j)_{j \in J}, (S_n)_{n \in \mathbb{N}} \in \mathcal{B}(X, Y)$.

1) If $(T_j)_{j \in J}$ is pointwise bounded then $(T_j)_{j \in J}$ is in fact uniformly bounded, i.e., $\sup_{j \in J} \|T_j\| < \infty$.

2) If $\sup_{j \in J} \|T_j\| = \infty$ then $\{x \in X : \sup_{j \in J} \|T_j\| < \infty\}$ is meager in $X$.

3) If $\lim_{n \to \infty} S_n(x) = S(x)$ exist for all $x \in X$ then $(S_n)_{n \in \mathbb{N}}$ is uniformly bounded and $S \in \mathcal{B}(X, Y)$ with $\|S\| \leq \sup_{n \in \mathbb{N}} \|S_n\|$.

**Proof.** 1) Step 1: We want to use Theorem 2.7 (Baire). The pointwise boundedness of $(T_j)_{j \in J}$ allows us to write

$$X = \bigcup_{k \in \mathbb{N}} \left\{ x \in X \mid \forall j \in J : \|T_j(x)\|_Y \leq k \right\} =: F_k$$

$$= \bigcup_{k \in \mathbb{N}} \bigcap_{j \in J} \left\{ x \in X \mid \|T_j(x)\|_Y \leq k \right\} =: F_{k,j}.$$

Since $T_j$ is continuous, $F_{k,j} = T_j^{-1}(B(0, k))$ is closed for all $j \in J$ and all $k \in \mathbb{N}$ and so is $F_k$ for all $k \in \mathbb{N}$. Because of $\emptyset \neq \text{Int}(X) = \text{Int}(\bigcup_{k \in \mathbb{N}} F_k)$, there exists $m \in \mathbb{N}$ such that $\text{Int}(F_m) \neq \emptyset$ since we get otherwise a contradiction to $X$ being a Baire space (see Theorem 2.6, item (2)). Since $\text{Int}(F_m)$ is open and nonempty, there exists $x \in F_m$ and $r > 0$ such that $B(x, r) \subseteq \text{Int}(F_m) \subset F_m$. By definition of $F_m$, we have $\|T_j(y)\|_Y \leq m$ for all $y \in B(x, r)$ and all $j \in J$. 
Step 2: Given \( z \in \overline{B(0,1)} \subseteq X \), the point \( \tilde{z} := x + \frac{r}{2}z \) lies in \( B(x,r) \). We rewrite \( z = \frac{1}{2}(\tilde{z} - x) \) and estimate

\[
\|T_j(z)\|_Y \leq \|T_j\left(\frac{2}{r}(\tilde{z} - x)\right)\|_Y = \frac{2}{r} \|T_j(\tilde{z}) - T_j(x)\|_Y \leq \frac{2}{r} \left(\|T_j(\tilde{z})\| + \|T_j(x)\|_Y\right) \leq \frac{4m}{r}
\]

for all \( j \in J \). Since the upper bound \( \frac{4m}{r} \) does not depend on \( j \), we obtain

\[
\sup_{j\in J}\|T_j\| = \sup_{j\in J}\{\|T_j(\tilde{z})\|_Y | \|z\| \leq 1\} \leq \frac{4m}{r} < \infty,
\]

i.e., \((T_j)_{j\in J}\) is uniformly bounded.

2) Using the notation from the proof or 1), we find

\[
X = \left\{ x \in X \left| \sup_{j\in J}\|T_j(x)\| = \infty \right\} \cup \left\{ x \in X \left| \sup_{j\in J}\|T_j(x)\| < \infty \right\} \right.
\]

\[
= \left\{ x \in X \left| \sup_{j\in J}\|T_j(x)\| = \infty \right\} \cup \bigcup_{k\in \mathbb{N}} F_k
\]

which is a countable union of closed sets. If now one of the \( F_k \) had nonempty interior then, in analogy to the proof of 1), we would obtain that \((T_j)_{j\in J}\) is uniformly bounded and in particular

\[
\left\{ x \in X \left| \sup_{j\in J}\|T_j(x)\| = \infty \right\} = \emptyset
\]

in contradiction to our assumption. Thus we must have \( \text{Int}(F_k) = \emptyset \) for all \( k \in \mathbb{N} \) implying

\[
\left\{ x \in X \left| \sup_{j\in J}\|T_j(x)\| < \infty \right\} \right. \bigcup \bigcup_{k\in \mathbb{N}} F_k
\]

to be meager.

3) We first show that \((S_n)_{n\in \mathbb{N}}\) is uniformly bounded, i.e., \( \sup_{n\in \mathbb{N}} \|S_n\| < \infty \). By assumption, the limit \( \lim_{n\to \infty} S_n(x) =: S(x) \) exists for all \( x \in X \). Therefore we have in particular \( \lim_{n\to \infty} \|S_n(x)\|_Y = \|S(x)\|_Y \) for all \( x \in X \) which in turn implies \( \sup_{n\in \mathbb{N}} \|S_n(x)\|_Y < \infty \) for all \( x \in X \), i.e., \((S_n)_{n\in \mathbb{N}}\) is pointwise bounded and, by item 1), uniformly bounded, i.e., \( \sup_{n\in \mathbb{N}} \|S_n\| < \infty \).

Now we show that \( S \) is a linear operator. We conclude for \( x, \tilde{x} \in X \) and \( \lambda \in \mathbb{R} \) that

\[
S(\lambda x + \tilde{x}) = \lim_{n\to \infty} S_n(\lambda x + \tilde{x}) = \lim_{n\to \infty}(\lambda S_n(x) + S_n(\tilde{x})) = \lambda \lim_{n\to \infty} S_n(x) + \lim_{n\to \infty} S_n(\tilde{x}) = \lambda S(x) + S(\tilde{x}).
\]
Now choose a constant \( C \geq \sup_{n \in \mathbb{N}} \| S_n \| \). Then boundedness of \( S \) follows now from the estimate
\[
\frac{\| S(x) \|_Y}{\| x \|_X} = \lim_{n \to \infty} \left\| S_n \left( \frac{x}{\| x \|_X} \right) \right\|_Y \leq \sup_{n \in \mathbb{N}} \left\| S_n \left( \frac{x}{\| x \|_X} \right) \right\|_Y \leq \sup_{n \in \mathbb{N}} \| S_n \| \leq C.
\]
□

For dual spaces, we obtain the following characterization of bounded sets.

**Corollary 2.18.** Let \((X, \| \cdot \|_X)\) be a Banach space and \((Y, \| \cdot \|_Y)\) a normed vector space. Then
\[
U \subseteq B(X, \mathbb{R}) \text{ bounded} \iff \sup_{f \in U} |f(x)| < \infty \quad \forall x \in X.
\]

**Proof.** ‘\(\Rightarrow\)’: Let \(U \subseteq B(X, \mathbb{R})\) be bounded, i.e., there exists \(C > 0\) such that \(\| f \| \leq C\) for all \(f \in U\). Therefore the inequality
\[
\frac{|f(x)|}{\| x \|_X} \leq \sup_{x \in X} \left| f \left( \frac{x}{\| x \|_X} \right) \right| = \| f \| \leq C
\]
is true for all \(f \in U\). As a consequence, we get \(|f(x)| \leq C \| x \|_X\) for all \(f \in U\) and all \(x \in X\). This implies \(\sup_{f \in U} |f(x)| \leq C \| x \|_X\) for all \(x \in X\), i.e., \(\sup_{f \in U} |f(x)| < \infty\) for all \(x \in X\).

‘\(\Leftarrow\)’: Since \(\mathbb{R}\) is Banach, so is \(B(X, \mathbb{R})\). The family \((f)_{f \in U} \subseteq B(X, \mathbb{R})\) is by assumption pointwise bounded. According to item 1) in Theorem 2.17 (Banach-Steinhaus), \((f)_{f \in U}\) is also uniformly bounded, i.e., \(\sup_{f \in U} \| f \| \leq C\) for some \(C > 0\), i.e., \(U\) is bounded in the operator norm. □

The following facts are worthwhile mentioning.

**Remark 2.19.** 1) In Theorem 2.17 (Banach-Steinhaus), the assumption, that \(X\) is Banach, is necessary and cannot be dropped.

2) There are also proofs of Theorem 2.17 (Banach-Steinhaus) that do not use Theorem 2.7 (Baire), as done for example in [Chacón & Rafeiro & Vallejo].

Theorem 2.17 (Banach-Steinhaus) has lots of applications within functional analysis and other areas of mathematics as we will see.

**Definition 2.20.** Let \(X_1, X_2, Y\) be vector spaces. The map \(L : X_1 \times X_2 \to Y\) is called a **bilinear operator** if, for all \(x_1 \in X_1\) and for all \(x_2 \in X_2\), the maps
\[
X_1 \to Y, \quad z_1 \mapsto L(z_1, x_2),
\]
\[
X_2 \to Y, \quad z_2 \mapsto L(x_1, z_2)
\]
are linear operators.
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Proposition 2.21 (Boundedness of bilinear operators). Let $X_1, X_2$ be Banach spaces and $L : X_1 \times X_2 \to \mathbb{R}$ a bilinear operator that is continuous in each variable. Then $L$ is continuous, i.e., given sequences $\alpha_n \to \alpha$ and $\beta_n \to \beta$, we have $L(\alpha_n, \beta_n) \to L(\alpha, \beta)$.

Proof. By linearity, it is enough to study continuity of $L$ at the origin. Let $(\alpha_n, \beta_n) \to (0, 0)$. We have to show that $L(\alpha_n, \beta_n) \to 0$. We define $T_n : X_2 \to \mathbb{R}$, $\eta \mapsto L(\alpha_n, \eta)$ which satisfies $(T_n)_{n \in \mathbb{N}} \in \mathcal{B}(X_2, \mathbb{R})$ and

$$
\lim_{n \to \infty} T_n(\eta) = \lim_{n \to \infty} L(\alpha_n, \eta) = 0 \quad \forall \eta \in X_2,
$$

meaning, for all $\eta \in X_2$, the pointwise limit $\lim_{n \to \infty} T_n(\eta) = 0$ exists. By item 3 of Theorem 2.17 (Banach-Steinhaus), we have $\sup_{n \in \mathbb{N}} \|T_n\| < \infty$. This implies the existence of $C > 0$ such that

$$
\frac{|L(\alpha_n, \eta)|}{\|\eta\|_{X_2}} = \frac{|T_n(\eta)|}{\|\eta\|_{X_2}} \leq C \quad \forall \eta \in X_2, \forall n \in \mathbb{N}.
$$

So we get in particular $|L(\alpha_n, \eta)| \leq C \|\eta\|_{X_2}$ and, in particular for $\beta_n$, the estimate $0 \leq |L(\alpha_n, \beta_n)| \leq C \|\beta_n\|_{X_2}$ which implies $L(\alpha_n, \beta_n) \to 0$ for $n \to \infty$.

Another interesting application lies within the roam of numerics when approximating integrals:

Given an interval $[a, b] \subset \mathbb{R}$ together with an $n$th subdivision

$$a =: t_0^{(n)} < t_1^{(n)} < \cdots < t_{n-1}^{(n)} < t_n^{(n)} := b$$

and associated weights $w_0^{(n)}, \ldots, w_n^{(n)}$, the following question arises: When does $\sum_{k=0}^{n} w_k^{(n)} f\left(t_k^{(n)}\right)$ converge to $\int_a^b f(t)dt$ for all $f \in C^0([a, b], \mathbb{R})$?

For $w_k^{(n)} = t_k^{(n)} - t_{k-1}^{(n)}$ we get the usual Riemann sums. Using Theorem 2.17 (Banach-Steinhaus), one can show more.

Proposition 2.22 (Szego). If both conditions

1) $\exists M > 0 : \sum_{k=0}^{n} |w_k^{(n)}| \leq M \quad \forall n \in \mathbb{N}$,

2) The sum $\sum_{k=0}^{n} w_k^{(n)} p\left(t_k^{(n)}\right)$ converges to $\int_a^b p(t)dt$ for $n \to \infty$ for all polynomials $p$, hold true then the sum $\sum_{k=0}^{n} w_k^{(n)} f\left(t_k^{(n)}\right)$ converges to $\int_a^b f(t)dt$ for $n \to \infty$ for all $f \in C^0([a, b], \mathbb{R})$.

Proof. Left to the reader. □
3. Open mapping, inverse mapping, and closed graph theorem

Given two topological spaces $X$ and $Y$, a continuous function $f : X \to Y$, and an open set $U \subseteq Y$, we know that the preimage $f^{-1}(U)$ is open. This motivates the following question. Under which assumptions on $X$, $Y$ and $f : X \to Y$ is the image of an open set open?

**Definition 2.23.** Let $X$, $Y$ be topological spaces. A map $f : X \to Y$ is said to be **open** if $f(U) \subseteq Y$ is open for all open $U \subseteq X$.

Let us look for examples of open maps.

**Example 2.24.** Let $X$, $Y$ be topological spaces and $f : X \to Y$ a homeomorphism. Then $f$ is open.

**Proof.** $f$ being a homeomorphism implies $f$ and $f^{-1}$ to be continuous. If $U \subseteq X$ is open, so is $(f^{-1})^{-1}(U) = f(U)$. □

Are continuous maps open?

**Example 2.25.** $f(x) := x^2$ is not open as map $f : \mathbb{R} \to \mathbb{R}$, only as map $f : \mathbb{R} \to \mathbb{R}^{\geq 0}$.

**Proof.** The set $f([-1, 1]) = [0, 1]$ is not open in $\mathbb{R}$. But in $\mathbb{R}^{\geq 0}$, sets of the form $[0, b]$ with $b > 0$ are open. □

Thus we conclude that continuous maps are not necessarily open. Can non-continuous maps be open? Recall the sign map

$$\text{sign} : \mathbb{R} \to \mathbb{R}, \quad \text{sign}(y) := \begin{cases} 1, & y > 0, \\ 0, & y = 0, \\ -1, & y < 0. \end{cases}$$

**Example 2.26.** The map $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) := x + \text{sign}(y)$ is noncontinuous but open.

**Proof.** The map $f$ is clearly not continuous. Recall that open 2-dimensional intervals of the form $]a_1, b_1[ \times ]a_2, b_2[ \subseteq \mathbb{R}^2$ generate the standard topology on $\mathbb{R}^2$. If such an 2-dimensional interval does not intersect the line $\mathbb{R} \times \{0\}$ its image under $f$ is of the form $]a_1 + 1, b_1 + 1[$, if $0 < a_2 < b_2$, or $]a_1 - 1, b_1 - 1[,$ if $a_2 < b_2 < 0$, and thus open. If $]a_1, b_1[ \times ]a_2, b_2[ \text{ intersects } \mathbb{R} \times \{0\}$ then its image is $]a_1 + 1, b_1 + 1[ \cup ]a_1, b_1[ \cup ]a_1 - 1, b_1 - 1[$, which is open. □

For continuous linear operators between Banach spaces, there exists the following important result.
**Theorem 2.27 (Open mapping I).** Let $X$ and $Y$ be Banach spaces and $T \in \mathcal{B}(X, Y)$ surjective. Then $T$ is open.

Before we start with the proof of Theorem 2.27 (Open mapping I), let us consider how balls behave under linear operators.

**Lemma 2.28.** Let $X$ and $Y$ be normed vector spaces and $T : X \to Y$ a linear operator.

1) $\forall x \in X, \forall r, s > 0$:

$$T(B(x, r)) = T(x) + T(B(0, r)) = T(x) + \frac{r}{s}T(B(0, s)).$$

2) $T(B(0, r))$ and $\overline{T(B(0, r))}$ are symmetric and convex for all $r > 0$. In particular, if they contain $B(y, s)$ for some $y \in Y$ and $s > 0$ then they contain also $B(-y, s)$ and $B(0, s) = \frac{1}{2}B(y, s) + \frac{1}{2}B(-y, s)$.

**Proof.** This follows from the linearity of $T$. \hfill □

We now are ready for

**Proof of Theorem 2.27 (Open mapping I).** 

**Step 1:** Given $U \subseteq X$, we have to show that $T(U) \subseteq Y$ is open. Since open sets in normed spaces can be written as union of open balls it is enough to show that the image of each open ball is open. By Lemma 2.28, it suffices to prove that $T(B(0, 1))$ is open, i.e., for all $y \in T(B(0, 1))$ there is $s > 0$ such that $B(y, s) \subseteq T(B(0, 1))$. By Lemma 2.28, it is enough to show this for $y = 0$.

**Step 2:** Since $T$ is surjective we can write $Y = \bigcup_{k \in \mathbb{N}} \overline{T(B(0, k))}$. As a Banach space, $Y$ is Baire and Theorem 2.6, item (2), implies that there must exist $n \in \mathbb{N}$ such that $\text{Int} \left( \overline{T(B(0, n))} \right) \neq \emptyset$. Since $\overline{T(B(0, n))} = n\overline{T(B(0, 1))}$, we also have $\text{Int} \left( \overline{T(B(0, 1))} \right) \neq \emptyset$. Thus there exists $y \in \text{Int} \left( \overline{T(B(0, 1))} \right)$ and $r > 0$ such that $B(y, r) \subseteq \text{Int} \left( \overline{T(B(0, 1))} \right) \subseteq \overline{T(B(0, 1))}$. By Lemma 2.28, we conclude $B(0, r) \subseteq \text{Int} \left( \overline{T(B(0, 1))} \right) \subseteq \overline{T(B(0, 1))}$ and $B \left( 0, \frac{r}{2^m} \right) \subseteq \overline{T \left( B \left( 0, \frac{1}{2^m} \right) \right)}$ for all $m \in \mathbb{N}$.

**Step 3:** We will show that $B \left( 0, \frac{1}{2} \right) \subseteq T(B(0, 1))$. For this purpose consider $z \in B \left( 0, \frac{1}{2} \right)$. Since $B \left( 0, \frac{1}{2} \right) \subseteq \overline{T \left( B \left( 0, \frac{1}{2} \right) \right)}$, there is $x_1 \in B \left( 0, \frac{1}{2} \right) \subseteq X$ such that $\|z - T(x_1)\|_Y < \frac{1}{2^2}$ implying $(z - T(x_1)) \in B \left( 0, \frac{1}{2^2} \right) \subseteq \overline{T \left( B \left( 0, \frac{1}{2^2} \right) \right)}$. Therefore there exists $x_2 \in B \left( 0, \frac{1}{2^2} \right)$ such that $\|z - T(x_1) - T(x_2)\|_Y < \frac{1}{2^3}$ implying $(z - T(x_1) - T(x_2)) \in B \left( 0, \frac{1}{2^3} \right) \subseteq \overline{T \left( B \left( 0, \frac{1}{2^3} \right) \right)}$. By iterating this procedure, we obtain a sequence $z_k := \left( z - \sum_{i=1}^{k} T(x_i) \right) \in B \left( 0, \frac{1}{2^k} \right) \subseteq \overline{T \left( B \left( 0, \frac{1}{2^k} \right) \right)}$ for all $k \in \mathbb{N}$. 

Since \( \|z_k\|_Y < \frac{r}{2^k} \) for all \( k \geq K - 1 \) the sequence \( (z_k)_{k \in \mathbb{N}} \) converges with limit \( 0 = \lim_{k \to \infty} z_k = z - \sum_{i=0}^{\infty} T(x_i) \) implying \( z = \sum_{i=0}^{\infty} T(x_i) \). Moreover, the estimate

\[
\left\| \sum_{i=1}^{k} x_i \right\|_X \leq \sum_{i=1}^{k} \|x_i\|_X < \sum_{i=1}^{k} \frac{1}{2^i} \leq 1
\]

implies that \( y_k := \sum_{i=1}^{k} x_i \) is a Cauchy sequence. Since \( X \) is Banach, \( (y_k)_{k \in \mathbb{N}} \) converges and \( \lim_{k \to \infty} y_k = y \) satisfies \( \|y\|_X < 1 \). Since \( T \) is continuous, we conclude

\[
T(y) = T \left( \lim_{k \to \infty} y_k \right) = \lim_{k \to \infty} T(y_k) = \lim_{k \to \infty} \sum_{i=1}^{k} T(x_i) = z
\]

implying \( B \left( 0, \frac{r}{2} \right) \subseteq T(B(0, 1)) \) since \( z \in B \left( 0, \frac{r}{2} \right) \) was chosen arbitrarily. \( \Box \)

An immediate result is the following statement.

**Theorem 2.29 (Inverse mapping).** Let \( X \) and \( Y \) be Banach spaces and \( T \in \mathcal{B}(X, Y) \) bijective. Then \( T^{-1} \in \mathcal{B}(Y, X) \).

**Proof.** By Theorem 2.27 (Open mapping I), \( T \) is open. This implies that the inverse map \( T^{-1} \) is continuous, i.e., \( T^{-1} \) is a bounded linear operator. \( \Box \)

It is worthwhile to mention that Theorem 2.27 (Open mapping I) can be reformulated as follows.

**Theorem 2.30 (Open mapping II).** Let \( X \) and \( Y \) be Banach spaces and \( T \in \mathcal{B}(X, Y) \) surjective. Then there exists \( c > 0 \) such that for all \( y \in Y \) there is \( x \in X \) with \( \|x\|_X \leq c \|y\|_Y \) and \( T(x) = y \).

**Proof.** See for example [Haase]. \( \Box \)

For linear operators, being bounded is the same as being continuous. Thus Theorem 2.30 (Open mapping II) intuitively measures how much the preimage \( x \in X \) of a given \( y \in Y \) can change if we vary \( y \). The change is controlled by \( c > 0 \). Theorem 2.30 (Open mapping II) does not say anything about how large \( c > 0 \) is. Thus the existence of \( c > 0 \) is of purely theoretical value and cannot be used for explicit estimates.

Many operators are only welldefined on subspaces of the spaces one would like to work on.

**Definition 2.31.** Let \( X \) and \( Y \) be normed vector spaces. If a linear operator \( T \) is only defined on a subspace of \( X \), we call this subspace the **domain** of \( T \) and denote it by \( \text{Dom}(T) \) and write

\[
T : \text{Dom}(T) \subseteq X \to Y.
\]
The set \( \{ T(x) \mid x \in \text{Dom}(T) \} \) is called the **range** or **image** of \( T \) and denoted by \( R(T) \) or \( \text{Im}(T) \).

That the domain does not coincide with the space one would like to work on, happens quite often.

**Example 2.32.** Consider \( X := \{ f \in C^0(\mathbb{R}, \mathbb{R}) \mid f \text{ bounded} \} \) with norm \( \| f \|_\infty := \sup_{x \in \mathbb{R}} |f(x)| \) and the differential operator \( T(f) := f' \). Then

1) For \( \text{Dom}(T) := \{ f \in C^1(\mathbb{R}, \mathbb{R}) \mid f' \text{ bounded} \} \), the operator \( T : \text{Dom}(T) \subset X \to X \) is well-defined.

2) \( T \) is not bounded.

**Proof.** If we want that \( \| f \|_\infty \) and \( \| T(f) \|_\infty = \| f' \|_\infty \) are finite then we have to choose

\[
\text{Dom}(T) = \{ f \in C^1(\mathbb{R}, \mathbb{R}) \mid f' \text{ bounded} \}.
\]

\( T \) is not bounded for the following reason. Consider the sequence \( (f_n)_{n \in \mathbb{N}} \) given by \( f_n(x) := \sin(nx) \) for which we get \( T(f_n)(x) = n \cos(nx) \). Then there is no \( C > 0 \) satisfying \( n = \| T(f_n) \|_\infty \leq C \| f_n \|_\infty = C \) for all \( n \in \mathbb{N} \). \( \square \)

The operator in Example 2.32 is an often used operator, so one may ask if there are ‘weaker’ properties than bounded, that it still satisfies.

**Definition 2.33.** Let \( X \) and \( Y \) be vector spaces and \( T : \text{Dom}(T) \subset X \to Y \) be a linear operator. \( T \) is **closed** if

\[
\text{graph}(T) := \{(x, T(x)) \mid x \in \text{Dom}(T)\}
\]

is closed in \( X \times Y \).

Reformulated, this means:

**Lemma 2.34.** Let \( X \) and \( Y \) be vector spaces and \( T : \text{Dom}(T) \subset X \to Y \) be a linear operator. Then \( T \) is closed if and only if for all sequences \( (x_n)_{n \in \mathbb{N}} \in \text{Dom}(T) \) and \( (y_n)_{n \in \mathbb{N}} := (T(x_n))_{n \in \mathbb{N}} \in Y \) with \( x_n \to x \) and \( y_n \to y \) follows that \( x \in \text{Dom}(T) \) and \( T(x) = y \).

**Proof.** Left to the reader. \( \square \)

Using this, we find the following.

**Example 2.35.** The differential operator in Example 2.32 is closed.

**Proof.** Let \( (f_n)_{n \in \mathbb{N}} \in \text{Dom}(T) = \{ h \in C^1(\mathbb{R}, \mathbb{R}) \mid h' \text{ bounded} \} \) and \( g_n := f_n' = T(f_n) \in \{ h \in C^0(\mathbb{R}, \mathbb{R}) \mid h \text{ bounded} \} \) and assume \( f_n \to f \) and \( g_n \to g \) in \( \| \|_\infty \) for some functions \( f \) and \( g \). Since \( \| \|_\infty \) imposes uniform convergence, the limits \( f \) and \( g \) are continuous and bounded and we have in particular \( f' = T(f) = g \), i.e., \( T \) is closed. \( \square \)
Thus closed operators are not necessarily bounded. Under which assumptions being closed and being bounded is the same, we will investigate now.

**Theorem 2.36 (Closed graph).** Let $X, Y$ be Banach spaces. Then we have $T \in \mathcal{B}(X, Y) \iff T$ closed.

**Proof.** ‘$\Rightarrow$’: Consider $\Phi : X \times Y \to \mathbb{R}$, $\Phi(x, y) := \|T(x) - y\|_Y$ which is continuous. Therefore $\text{graph}(T) = \Phi^{-1}(\{0\})$ is closed.

‘$\Leftarrow$’: Consider $(X \times Y, \|\|)$ with $\|(x, y)\| := \|x\|_X + \|y\|_Y$ for all $x \in X$ and $y \in Y$ which is a Banach space. Let $\text{graph}(T)$ be closed. As closed subset of $(X \times Y, \|\|)$, it is itself a Banach space. We consider the projections

\[
\pi_1 : \text{graph}(T) \to X, \quad \pi_1(x, T(x)) = x,
\]

\[
\pi_2 : \text{graph}(T) \to Y, \quad \pi_2(x, T(x)) = T(x).
\]

Since

\[
\|(x, T(x))\| = \|x\|_X + \|T(x)\|_Y \geq \|x\|_X = \|\pi_1(x, T(x))\|_X,
\]

\[
\|(x, T(x))\| = \|x\|_X + \|T(x)\|_Y \geq \|T(x)\|_Y = \|\pi_2(x, T(x))\|_Y,
\]

$\pi_1$ and $\pi_2$ are continuous. $\pi_1$ is bijective such that $\pi_1^{-1} \in \mathcal{B}(\text{graph}(T), X)$ by Theorem 2.29 (Inverse mapping). Thus $T = \pi_2 \circ \pi_1^{-1}$ is continuous. □

In fact, Theorem 2.27 (Open mapping I), Theorem 2.29 (Inverse mapping), and Theorem 2.36 (Closed graph) are all equivalent. A nice application is the following statement.

**Proposition 2.37 (Hellinger-Toeplitz).** Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $T : H \to H$ a linear operator with

\[
\langle T(x), y \rangle = \langle x, T(y) \rangle \quad \text{for all } x, y \in H.
\]

Then $T$ is bounded.

**Proof.** Left to the reader. □

### 4. Seminorms and Fréchet spaces

There are many spaces where the ‘natural norm’ is not well defined or finite, like when passing from $(C^0([0, 1], \mathbb{R}), \|\|_\infty)$ to $C^0(]0, 1[\mathbb{R})$ where the supremum norm $\|\|_\infty$ is not finite. Just picking a compact interval $[a, b] \subset ]0, 1[$ and setting

\[
p_{[a,b]}(f) := \max_{x \in [a,b]} |f(x)|
\]

for functions $f \in C^0(]0, 1[\mathbb{R})$ does not solve the problem since $p_{[a,b]}(f) = 0$ does not imply that $f \equiv 0$ on the whole interval $]0, 1[$. The question now is, if exhausting $]0, 1[$ by compact intervals $[a, b] \subset ]0, 1[$ may do the trick and, if yes, if this approach always works in such situations.
Definition 2.38. Let $X$ be a vector space over $K \in \{\mathbb{R}, \mathbb{C}\}$. The function $p : X \to \mathbb{R}$ is a seminorm if

1) $p(x) \geq 0$ for all $x \in X$.
2) $p(\lambda x) = |\lambda| p(x)$ for all $x \in X$ and all $\lambda \in K$.
3) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$.

A seminorm $p$ is no norm since $p(x) = 0$ does not imply $x = 0$. For the same reason, setting $d(x, y) := p(x - y)$ does not necessarily lead to a metric since $d(x, y) = 0$ does not necessarily imply $x = y$. Nevertheless notice that $d$ satisfies $d(x + v, y + v) = d(x, y)$ for all $x, y, v \in X$, i.e., $d$ is translation invariant. To obtain a metric, we need additional properties.

Definition 2.39. A sequence of seminorms $(p_k)_{k \in \mathbb{N}}$ on a vector space $X$ is separating if for all $x \in X \setminus \{0\}$ there exists $k \in \mathbb{N}$ such that $p_k(x) > 0$.

Now we can actually construct a metric.

Lemma 2.40. Let $(p_k)_{k \in \mathbb{N}}$ be a separating sequence of seminorms on the vector space $X$. Then

$$d : X \times X \to \mathbb{R}_{\geq 0}, \quad d(x, y) := \sum_{k=1}^{\infty} \frac{p_k(x - y)}{2^k(1 + p_k(x - y))}$$

is a translation invariant metric on $X$.

Proof. To prove the triangle inequality note that the function $h : \mathbb{R}_{\geq 0} \to \mathbb{R}, \quad h(z) := \frac{1}{1+z}$ is strictly increasing. The rest are easy verifications left to the reader. $\square$

The following type of spaces is named after the French mathematician Maurice Fréchet (1878 – 1973).

Definition 2.41. A space with a translation invariant, complete metric is said to be a Fréchet space.

To avoid confusion, let us remark the following.

Remark 2.42. The translation invariant, complete metric of a Fréchet space is usually not induced by a norm. A Banach space can be made into a Fréchet space by endowing it with the translation invariant, complete metric induced by its complete norm.

Before we consider examples, we need some notation.

Definition 2.43. Let $U, V \subseteq \mathbb{R}^n$ be open. $U$ is compactly contained in $V$, written $U \subset\subset V$ if the closure $\overline{U}$ is a compact subset of $V$.

Let us consider two standard examples.
Example 2.44. Let $\Omega \subseteq \mathbb{R}^n$ be open.

1) $C^0(\Omega, \mathbb{R})$ can be endowed with the structure of a Fréchet space.
2) $L^p_{\text{loc}}(\Omega) := \left\{ f : \Omega \to \mathbb{R} \mid f \text{ measurable}, \int_U |f(x)|^p \, dx < \infty \text{ for all } U \subset \subset \Omega \right\}$ can be endowed with the structure of a Fréchet space for all $p \in [1, \infty]$.

Proof. 1) Given $\Omega \subseteq \mathbb{R}^n$ open, we define

$$U_k := \left\{ x \in \Omega \mid \|x\|_{\text{eucl}} \leq k, \ d_{\text{eucl}}(x, \partial \Omega) \geq \frac{1}{k} \right\}$$

and set $p_k(f) := \max_{x \in U_k} |f(x)|$ for functions $f \in C^0(\Omega, \mathbb{R})$. The sequence $(p_k)_{k \in \mathbb{N}}$ is a separating sequence of seminorms. Using Lemma 2.40, we get a complete metric.

2) We use the sets $U_k$ from (2.45) to define for functions $f \in L^p_{\text{loc}}(\Omega)$ the seminorm

$$p_k(f) := \|f\|_{L^p(U_k)} := \left( \int_{U_k} |f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

The sequence $(p_k)_{k \in \mathbb{N}}$ is a separating sequence of seminorms. Using Lemma 2.40, we get a complete metric. \qed

5. The Hahn-Banach extension and separation theorems

This section studies the questions if it is possible to extend linear maps from a subspace to the whole vector space and, if yes, under which conditions this can be done, which properties the extension has and if the extension is unique or not. Moreover, this concept can be used to find ‘separating hyperplanes’ between two nonempty, disjoint, convex subsets. We start with the following notation.

Definition 2.46. Let $X$ be a vector space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A linear map $f : X \to \mathbb{K}$ is often called a functional.

In order to extend a map, it turns out that it has to satisfy certain growth conditions which we will formulate by means of the following type of functions.

Definition 2.47. Let $X$ be a vector space over $\mathbb{R}$. A function $p : X \to \mathbb{R}$ is called Hahn-Banach functional if

1) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$.
2) $p(tx) = tp(x)$ for all $t \geq 0$.

Hans Hahn (1879 – 1934) was an Austrian mathematician. Note that Hahn-Banach functionals may have negative values, but behave otherwise quite
like norms. Moreover, Hahn-Banach functionals are convex since we have for all $\lambda \in ]0, 1[$ and $x, y \in X$

$$p(\lambda x + (1 - \lambda)y) \leq p(\lambda x) + p((1 - \lambda)y) = \lambda p(x) + (1 - \lambda)p(y).$$

**Example 2.48.**

1) Norms and seminorms are Hahn-Banach functionals.

2) Let $X$ be a normed vector space, $\Omega \subset X$ open, bounded, convex with $0 \in \Omega$. Then the so-called **Minkowski functional** of $\Omega$ given by

$$p_\Omega : X \to \mathbb{R}_{\geq 0}, \quad p_\Omega(x) := \inf\{\lambda \geq 0 | x \in \lambda\Omega\}$$

is a Hahn-Banach functional.

Hermann Minkowski (1864 – 1909) was a German mathematician. The Minkowski functional $p_\Omega(x)$ measures how much one may scale $\Omega$ in order to have $x$ ‘just’ sitting in the scaled set $\lambda\Omega$. Now we come to the central statement of this section.

**Theorem 2.49 (Hahn-Banach).** Let $X$ be an $\mathbb{R}$-vector space and $p : X \to \mathbb{R}$ a Hahn-Banach functional. Let $V \subseteq X$ be a subspace and $f : V \to \mathbb{R}$ a linear functional satisfying $f(v) \leq p(v)$ for all $v \in V$. Then there exists a linear functional $F : X \to \mathbb{R}$ satisfying

$$F|_V = f \quad \text{and} \quad -p(-x) \leq F(x) \leq p(x) \quad \forall x \in X.$$

$F(x) \leq p(x)$ for all $x \in X$ implies immediately $F(-x) \leq p(-x)$ and thus, by multiplying with $(-1)$ and linearity, we get $-p(-x) \leq F(x)$. Therefore, in the literature, one often finds only the statement $F(x) \leq p(x)$. But for applications, it is useful to consider $F(x)$ as ‘sandwiched’ between $p(x)$ and $-p(-x)$.

**Proof of Theorem 2.49 (Hahn-Banach).** If $V = X$ then we can choose $F := f$ since $f(x) = -f(-x) \geq -p(-x)$. Thus let us assume from now on that $V \subset X$ with $V \neq X$.

**Step 1:** Pick $x_0 \in X \setminus V$ and consider

$$\text{Span}_\mathbb{R}\{V, x_0\} = \{v + tx_0 | v \in V, \ t \in \mathbb{R}\} =: V_0.$$

Then we have $V \subset V_0$ with $V \neq V_0$. Now we want to find some kind of ‘normalized maximal distance’ between $f$ and $p$. For all $u, v \in V$, we get the estimate

$$f(u) + f(v) = f(u + v) \leq p(u + v) = p(u - x_0 + x_0 + v) \leq p(u - x_0) + p(x_0 + v)$$

which we reformulate as

$$f(u) - p(u - x_0) \leq p(v + x_0) - f(v) \quad \forall u, v \in V.$$
This yields for all \( u, v \in V \)
\[
(2.50) \quad f(u) - p(u - x_0) \leq \sup_{u \in V} \{ f(u) - p(u - x_0) \} \leq p(v + x_0) - f(v) \leq \beta
\]

**Step 2:** Now we want to extend \( f \) to \( V_0 \). Our candidate is
\[
f_0 : V_0 \to \mathbb{R}, \quad f_0(u + tx_0) := f(u) + t\beta
\]
which clearly satisfies \( f_0|_V = f \). We now show that \( f_0 \leq p \) in \( V_0 \), i.e., we have to show
\[
f_0(u + tx_0) \leq p(u + tx_0) \quad \forall u \in V, \forall t \in \mathbb{R}
\]
which is trivial for \( t = 0 \). Consider now \( t > 0 \), replace \( u \) and \( v \) in (2.50) by \( \frac{u}{t} \), and multiply by \( t \). This yields
\[
t \left( f \left( \frac{u}{t} \right) - p \left( \frac{u}{t} - x_0 \right) \right) \leq t\beta \leq t \left( p \left( \frac{u}{t} + x_0 \right) - f \left( \frac{u}{t} \right) \right)
\]
which in turn yields
\[
f(u) - p(u - tx_0) \leq t\beta \leq p(u + tx_0) - f(u).
\]
By definition of \( f_0 \), we get
\[
f_0(u + tx_0) = f(u) + t\beta \leq p(u + tx_0),
\]
\[
f_0(u - tx_0) = f(u) - t\beta \leq p(u - tx_0)
\]
such that \( f_0(u + sx_0) \leq p(u + sx_0) \) for all \( u \in U \) and \( s \in \mathbb{R} \), i.e., on the whole space \( V_0 \).

**Step 3:** The idea is to repeat Step 2 until the extended subspace coincides with \( X \). More precisely, consider the set
\[
\mathcal{U} := \left\{ (U, f_U) \mid V \subseteq U \subseteq X \text{ subspace}, \begin{array}{l}
f_U : U \to \mathbb{R} \text{ linear}, f_U|_V = f, \\
f_U(u) \leq p(u) \forall u \in U.
\end{array} \right\}
\]
carrying the partial order \( \preceq \) given by
\[
(U, f_U) \preceq (W, f_W) \iff U \subseteq W \text{ and } f_W|_U = f_U.
\]
By Theorem A.12 (Hausdorff maximality principle), \( \mathcal{U} \) has a maximal element \((U_{\max}, f_{U_{\max}})\). We must have \( U_{\max} = X \) since otherwise, if \( U_{\max} \) is strictly smaller than \( X \), we could extend it by Step 2 in contradiction to its maximality. Moreover,
\[
F := f_{U_{\max}} : X \to \mathbb{R}
\]
is linear and \( F|_V = f \) and \( F(x) \leq p(x) \) for all \( x \in X \) by construction. \( \square \)

Hahn-Banach extensions are ‘compatible’ with boundedness:
**Theorem 2.51 (Hahn-Banach for bounded operators).** Let \((X, \|\|)\) be a normed vector space over \(K \in \{\mathbb{R}, \mathbb{C}\}\) and \(V \subseteq X\) a subspace and \(f \in \mathcal{B}(V, K)\). Then there is \(F \in \mathcal{B}(X, K)\) with \(F|_V = f\) and coinciding operator norms \(\|F\|_o = \|f\|_o\).

**Proof.** Case \(K = \mathbb{R}\): Define \(p(x) := \|x\|\|f\|_o\) which is a Hahn-Banach functional. Thus there is \(F : X \rightarrow \mathbb{R}\) with \(F|_V = f\) and

\[-\|x\|\|f\|_o = -p(-x) \leq F(x) \leq p(x) = \|x\|\|f\|_o\]

which implies \(\|F\|_o \leq \|f\|_o\). Since \(F : X \rightarrow \mathbb{R}\) is an extension of \(f : V \rightarrow \mathbb{R}\) we automatically have also \(\|F\|_o \geq \|f\|_o\).

Case \(K = \mathbb{C}\): We decompose the \(\mathbb{C}\)-linear functional \(f : V \rightarrow \mathbb{C}\) into \(f = \Re(f) + i\Im(f)\) with \(\mathbb{R}\)-linear real and imaginary parts \(\Re(f), \Im(f) : V \rightarrow \mathbb{R}\). We automatically have \(\|\Re(f)\|_o \leq \|f\|_o\). Using the already proven case \(K = \mathbb{R}\), we get an \(\mathbb{R}\)-linear function \(\Phi : X \rightarrow \mathbb{R}\) with \(\Phi|_V = \Re(f)\) and \(\|\Re(f)\|_o = \|\Phi\|_o \leq \|f\|_o\). Then we define

\[F : X \rightarrow \mathbb{C}, \quad F(x) := \Phi(x) - i\Phi(ix)\]

A short calculation shows \(\Re(f)(iv) = -\Im(f)\) for all \(v \in V\) such that we get

\[F(v) = \Phi(v) - i\Phi(iv) = \Re(f)(v) - i\Re(f)(iv) = \Re(f)(v) + i\Im(f)(v) = f(v)\]

for all \(v \in V\). Now write

\[F(x) = |F(x)| \frac{F(x)}{|F(x)|} = |F(x)| \lambda_F(x)\]

with \(\lambda_F(x) \in \mathbb{C}\) and \(|\lambda_F(x)| = 1\). Then we have

\[|F(x)| = (\lambda_F(x))^{-1} F(x) = F\left((\lambda_F(x))^{-1} x\right) = \Phi\left((\lambda_F(x))^{-1} x\right)\]

\[\leq \|\Phi\|_o \|\lambda_F(x)^{-1} x\| = \|\Phi\|_o \|x\|\]

implying \(\|F\|_o \leq \|f\|_o\). Since \(F : X \rightarrow \mathbb{C}\) is an extension of \(f : V \rightarrow \mathbb{C}\) we automatically have \(\|F\|_o \geq \|f\|_o\), too. \[\square\]

Theorem 2.51 (Hahn-Banach) implies the following useful facts.

**Corollary 2.52.** Let \(K \in \{\mathbb{R}, \mathbb{C}\}\) and let \(X\) be a normed \(K\)-vector space. Let \(x, y \in X\) with \(x \neq y\). Then there is \(F \in \mathcal{B}(X, K)\) with \(F(x) \neq F(y)\).

**Proof.** \(x, y \in X\) with \(x \neq y\) implies \(z := (x - y) \neq 0\). Consider

\[\text{Span}_K z = \{\lambda z \mid \lambda \in K\} =: V\]
and \( f : V \to \mathbb{K} \) given by \( f(\lambda x) := \lambda \) which implies \( f(x - y) = f(z) = 1 \). By Theorem 2.51 (Hahn-Banach), there is \( F \in \mathcal{B}(X, \mathbb{K}) \) with \( F|_V = f \) and we get

\[
F(x) - F(y) = F(x - y) = f(x - y) = 1
\]

such that \( F(x) = F(y) + 1 \) implying \( F(x) \neq F(y) \).

Later on, the following statement proves to be very useful.

**Corollary 2.53.** Let \( \mathbb{K} \in \{ \mathbb{R}, \mathbb{C} \} \), let \( (X, \| \cdot \|) \) be a normed \( \mathbb{K} \)-vector space, and let \( x \in X \). Then there is \( F \in \mathcal{B}(X, \mathbb{K}) \) with operator norm \( \|F\|_{op} = 1 \) and \( F(x) = \|x\| \).

**Proof.** Let \( x \in X \) and define \( V := \text{Span}_\mathbb{K}\{x\} = \{\lambda x \mid \lambda \in \mathbb{K}\} \) and the \( \mathbb{K} \)-linear map \( f : V \to \mathbb{R}, f(\lambda x) := \lambda \|x\| \). We get \( f(x) = \|x\| \) and

\[
\frac{|f(\lambda x)|}{\|\lambda x\|} = \frac{|\lambda| \|x\|}{|\lambda| \|x\|} = 1
\]

implying \( \|f\|_{op} = 1 \). By Theorem 2.51 (Hahn-Banach), there is \( F : X \to \mathbb{K} \) with \( F|_V = f \) and \( \|F\|_{op} = \|f\|_{op} = 1 \). Moreover, we have \( F(x) = f(x) = \|x\| \).

This we also will need later on:

**Corollary 2.54.** Let \( \mathbb{K} \in \{ \mathbb{R}, \mathbb{C} \} \) and let \( (X, \| \cdot \|) \) be a normed \( \mathbb{K} \)-vector space. Moreover, let \( Y \subset X \) be a closed subspace with \( x_0 \in X \setminus Y \). Then there exists \( f \in X^* \) such that \( f|_V = 0, \|f\|_{op} = 1 \), and \( f(x_0) = \text{dist}(x_0, Y) \).

**Proof.** Cf. for instance [Alt].

Given two sets \( U \) and \( V \) with \( U \cap V = \emptyset \), a natural question to ask is ‘how much’ these sets are disjoint, i.e., can we find a set \( W \) with \( W \cap U = \emptyset = W \cap V \) separating \( U \) from \( V \)? If yes, what properties and shape may \( W \) have?

Since we are working in a linear setting in this chapter, it makes sense to require of a separating set \( W \) to be ‘compatible’ in some way with linearity, meaning for example, \( W \) being a subspace of codimension 1. This requirement in turn imposes restrictions on the possible shapes of \( U \) and \( V \), i.e., they also have to be ‘compatible’ with linearity in some way.

**Definition 2.55.** Let \( X \) be a vector space and \( X \neq \emptyset \).

1) A **hyperplane** is a subspace \( V \subset X \) with \( V \neq X \) such that, for all subspaces \( U \) with \( V \subset U \subset X \), it follows either \( U = V \) or \( U = X \).

2) An **affine hyperplane** is a subset of the form \( V + x = \{v + x \mid v \in V\} \) where \( V \subset X \) is a hyperplane and \( x \in X \) a point.

Hyperplanes in possibly infinite vector spaces are characterized similarly as hyperplanes in finite dimensions:
Remark 2.56. Let $X$ be a vector space and $X \neq \{0\}$.
1) $V \subset X$ is a hyperplane if and only if there exists a nonvanishing linear functional $f : X \to \mathbb{R}$ such that $V = f^{-1}(0)$.
2) $V \subset X$ is an affine hyperplane if and only if there exist a nonvanishing linear functional $f : X \to \mathbb{R}$ and $r \in \mathbb{R}$ such that $V = f^{-1}(r)$.

If we want to separate two disjoint, convex sets the following theorem describes possible separating sets.

Theorem 2.57 (Separation of convex sets). Let $(X, \| \|)$ be a real normed vector space and $U, V \subset X$ nonempty subsets that are disjoint and convex.

1) If $U$ is open then there exist $F \in \mathcal{B}(X, \mathbb{R})$ and $c \in \mathbb{R}$ such that $F(u) < c \leq F(v) \quad \forall u \in U, \forall v \in V$.

2) If $U$ is compact and $V$ closed then there exist $F \in \mathcal{B}(X, \mathbb{R})$ and two real numbers $c_1 < c_2$ such that $F(u) \leq c_1 < c_2 \leq F(v) \quad \forall u \in U, \forall v \in V$.

Proof. 1) Step 1: In order to apply Theorem 2.49 (Hahn-Banach), we will construct a Hahn-Banach functional. Pick $u_0 \in U$ and $v_0 \in V$, set $x_0 := v_0 - u_0$, and define

$$\Omega := U - V + x_0 = \{(u - u_0) + (v_0 - v) \mid u \in U, v \in V\}.$$ 

One verifies easily that $U - V$ is convex and therefore $U - V + x_0 = \Omega$, too. Writing $\Omega = (\bigcup_{v \in V} (U - v)) + x_0$, we see that $\Omega$ is open. Moreover, we have $0 \in \Omega$ and $x_0 \neq \Omega$. The Minkowski functional $p_\Omega : X \to \mathbb{R}_{\geq 0}$ of $\Omega$ is given by

$$p_\Omega(x) = \inf\{\lambda \geq 0 \mid x \in \lambda \Omega\}.$$ 

According to Example 2.48, it is a Hahn-Banach functional, i.e., we have in particular for all $x, y \in X$ and $t \geq 0$,

$$p_\Omega(x + y) \leq p_\Omega(x) + p_\Omega(y) \quad \text{and} \quad p(tx) = tp(x).$$

Since $0 \in \Omega$ and since $\Omega$ is open, there exists $r > 0$ such that $B(0, r) \subseteq \Omega$ and we get the estimate

$$p_\Omega(x) \leq \frac{\|x\|}{r} \quad \forall x \in X.$$ 

Note that, since $x_0 \notin \Omega$, we have $p_\Omega(x_0) \geq 1$.

Step 2: Now we construct the map that we want to extend by means of Theorem 2.49 (Hahn-Banach). Set $Y := \text{Span}_\mathbb{R}\{x_0\} = \{tx_0 \mid t \in \mathbb{R}\}$ and define $f : Y \to \mathbb{R}$ by $f(tx_0) := t$. This linear functional satisfies $f(x_0) = 1$ and

$$f(tx_0) = t \leq tp_\Omega(x_0) = p(tx_0).$$
By Theorem 2.49 (Hahn-Banach), there exists a linear functional $F : X \to \mathbb{R}$ with $F|_Y = f$ and $-p_\Omega(-x) \leq F(x) \leq p(x)$ for all $x \in X$. The estimate
\[
\frac{|F(x)|}{\|x\|} \leq \frac{|p(x)|}{\|x\|} \leq \frac{\|x\|}{r \|x\|} = \frac{1}{r}
\]
implies $\|F\|_{op} \leq \frac{1}{r}$ such that $F$ is bounded.

**Step 3:** Given $u \in U$ and $v \in V$, the points $u - v + x_0$ lies in $\Omega$ which is open implying $p_\Omega(u - v + x_0) < 1$. Thus we get
\[
F(u) - F(v) + 1 = F(u - v + x_0) \leq p_\Omega(u - v + x_0) < 1
\]
and thus $F(u) < F(v)$ for all $u \in U$ and $v \in V$ implying $F(U) \cap F(V) = \emptyset$. Since $U$ and $V$ are convex, so are $F(U)$ and $F(V)$. In particular, $F(U)$ and $F(V)$ are connected and form thus intervals in $\mathbb{R}$. Since $F$ is bounded and $U$ open the range $F(U)$ is by Theorem 2.27 (Open mapping I) an open interval, i.e., not containing its upper boundary point $c := \sup_{u \in U} F(u)$. Thus we have $F(u) < c \leq F(v)$ for all $u \in U$ and $v \in V$.

2) Since $U$ is compact, $V$ closed and $U \cap V = \emptyset$, the distance between $U$ and $V$
\[
r := \inf\{\|u - v\| \mid u \in U, \ v \in V\}
\]
is strictly positive. The collar neighbourhood
\[
U_r := \{x \in X \mid \exists u \in U : \|x - u\| < r\}
\]
of $U$ is open and convex and satisfies $U_r \cap V = \emptyset$. Applying part 1) to $U_r$ and $V$, we obtain a bounded functional $F : X \to \mathbb{R}$ and the existence of a constant $c_2 > 0$ such that
\[
F(u) < c_2 \leq F(v) \quad \forall u \in U_r, \ \forall v \in V.
\]
Since $U$ is compact and $F$ continuous the range $F(U) \subset \mathbb{R}$ is compact containing its upper boundary point $c_1 := \sup_{u \in U} F(u)$. Thus we obtain $F(u) \leq c_1 < c_2 \leq F(v)$ for all $u \in U$ and $v \in V$. \hfill \Box

So far, we only addressed existence of Hahn-Banach extensions. Let us now investigate under which assumptions Hahn-Banach extensions are unique.

**Definition 2.58.** Let $(X, \|\|)$ be a normed vector space. $\mathcal{B}(X, \mathbb{R})$ is strictly convex if its closed unit ball is strictly convex, i.e., for all $f, g \in \mathcal{B}(X, \mathbb{R})$ with operator norm $\|f\|_{op} = 1 = \|g\|_{op}$ and all $\lambda \in [0, 1]$, we have
\[
\|\lambda f + (1 - \lambda)g\|_{op} < \lambda \|f\|_{op} + (1 - \lambda)\|g\|_{op} = 1.
\]
This is equivalent with requiring for $\mathcal{B}(X, \mathbb{R})$ that the closed ball centered at the origin of radius $r > 0$ is strictly convex for one and, by linearity, thus any $r > 0$.

**Theorem 2.59.** Let $(X, \|\|)$ be a normed space such that $\mathcal{B}(X, \mathbb{R})$ is strictly convex. Then the extension in Theorem 2.51 (Hahn-Banach) is unique.
Proof. Let $V \subseteq X$ be a subspace with $V \neq X$ and let $f \in \mathcal{B}(V, \mathbb{R})$. Now assume that, when applying Theorem 2.51 (Hahn-Banach), we obtain two different $F_0, F_1 \in \mathcal{B}(X, \mathbb{R})$ satisfying $F_0|_V = F_1|_V = f$ with coinciding operator norms $\|F_0\|_{\text{op}} = ||f||_{\text{op}} = ||F_1||_{\text{op}}$. W.l.o.g. we can assume by linearity and rescaling that $||f||_{\text{op}} = ||F_0||_{\text{op}} = ||F_1||_{\text{op}} = 1$. Let $\lambda \in ]0, 1[$ and define

$$F_\lambda := \lambda F_0 + (1 - \lambda) F_1.$$ 

Since $F_\lambda|_V = \lambda F_0|_V + (1 - \lambda) F_1|_V = \lambda f + (1 - \lambda) f = f$, the functional $F_\lambda$ is also an extension of $f$, thus we have automatically $\|F_\lambda\|_{\text{op}} \geq ||f||_{\text{op}}$.

Moreover, we estimate

$$||F_\lambda||_{\text{op}} = \|\lambda F_0 + (1 - \lambda) F_1\|_{\text{op}} \leq \lambda \|F_0\| + (1 - \lambda) \|F_1\|_{\text{op}} = 1 = ||f||_{\text{op}},$$

implies $\|F_\lambda\| = ||f||_{\text{op}} = 1$. This leads to a contradiction since $\|F_\lambda\|_{\text{op}} < 1$ according to the strict convexity of $\mathcal{B}(X, \mathbb{R})$. \qed
CHAPTER 3

Weak and weak* convergence and compactness

Banach spaces can be endowed with different types of convergences. In this section, we define and study the three most common ones, namely the norm convergence, the weak convergence, and the weak* convergence. We are mainly interested in their compactness properties.

1. Strong, weak, and weak* convergence and reflexive spaces

If not declared otherwise, \((X, \|\|)\) is a Banach space over \(\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}\) throughout this paragraph.

**Definition 3.1.** Given a Banach space \((X, \|\|)\), the space \(X^* := \mathcal{B}(X, \mathbb{K})\) is said to be the dual space of \(X\). We denote the operator norm on \(X^*\) by \(\|\|_*\).

In the literature, the notation \(X'\) for the dual space is also common.

**Remark 3.2.** Since \((\mathbb{K}, \|\|)\) is Banach, \((X^*, \|\|_*)\) is a Banach space.

There are several ‘natural’ types of convergence on \(X\) and \(X^*\):

**Definition 3.3.** Let \((X, \|\|)\) be a Banach space, \((X^*, \|\|_*)\) its dual, and \((x_n)_{n \in \mathbb{N}} \in X\) and \((x^*_n)_{n \in \mathbb{N}} \in X^*\).

1) The sequence \((x_n)_{n \in \mathbb{N}}\) converges strongly or in norm to \(x \in X\), in symbols \(x_n \to x\), if \(\lim_{n \to \infty} \|x_n - x\| = 0\).

2) The sequence \((x^*_n)_{n \in \mathbb{N}}\) converges strongly or in norm to \(x^* \in X^*\), in symbols \(x^*_n \to x^*\), if \(\lim_{n \to \infty} \|x^*_n - x^*\|_* = 0\).

3) The sequence \((x_n)_{n \in \mathbb{N}}\) converges weakly to \(x \in X\), in symbols \(x_n \rightharpoonup x\) or \(x_n \wto x\), if \(\lim_{n \to \infty} f(x_n) = f(x)\) for all \(f \in X^*\). In this situation, \(x\) is called the weak limit of \((x_n)_{n \in \mathbb{N}}\).

4) The sequence \((x^*_n)_{n \in \mathbb{N}}\) converges weakly* to \(x^* \in X^*\), in symbols \(x^*_n \rightharpoonup x^*\) or \(x^*_n \rightharpoonup x^*\), if \(\lim_{n \to \infty} x^*_n(z) = x^*(z)\) for all \(z \in X\). In this situation, \(x^*\) is called the weak* limit of \((x^*_n)_{n \in \mathbb{N}}\).

We observe the following relation between these types of convergence.

**Proposition 3.4.**

1) Strong convergence on \(X\) implies weak convergence on \(X\).

2) Strong convergence on \(X^*\) implies weak* convergence on \(X^*\).
3) **Weak and weak* limits are unique.**

**Proof.**

1) Let \( x_n \to x \) in \( X \). By definition, all \( f \in X^* \) are continuous. This implies \( \lim_{n \to \infty} f(x_n) = f(x) \) for all \( f \in X^* \), i.e., \( x_n \to x \).

2) \( x_n^* \to x^* \) in \( X^* \) means

\[
0 = \lim_{n \to \infty} \left\| x_n^* - x^* \right\|_* = \lim_{n \to \infty} \sup \left\{ \left| x_n^*(z) - x^*(z) \right| \mid z \in X, \left\| z \right\| \leq 1 \right\},
\]
i.e., for all \( z \in X \), we have by linearity

\[
0 = \lim_{n \to \infty} \left| x_n^* \left( \frac{z}{\left\| z \right\|} \right) - x^* \left( \frac{z}{\left\| z \right\|} \right) \right| = \lim_{n \to \infty} \left| x_n^*(z) - x^*(z) \right| = \frac{1}{\left\| z \right\|} \lim_{n \to \infty} \left| x_n^*(z) - x^*(z) \right|
\]
implying \( x_n^* \xrightarrow{w^*} x^* \).

3) The weak* limit is clearly unique. Now let us assume that the weak limit is not unique, i.e., assume there are \( x, y \in X \) with \( x \neq y \) but \( x_n \to x \) and \( x_n \to y \). According to Corollary 2.52, there exists \( f \in X^* \) with \( f(x) \neq f(y) \). But the definition of weakly convergent implies \( f(y) = \lim_{n \to \infty} f(x_n) = f(x) \) which contradicts \( f(x) \neq f(y) \). \( \square \)

Of course, we can iterate ‘dualizing’ spaces. The dual of the dual space is of special interest:

**Definition 3.5.** Let \((X, ||||)\) be a Banach space. The space \( X^{**} := (X^*)^* \) endowed with its operator norm \( ||| |||_* \) is called the **bidual space of** \( X \).

The bidual space \((X^{**}, ||| |||_*)\) is a Banach space. We are interested in it for the following reason.

**Proposition 3.6.** The map

\[
J : (X, |||) \to (X^{**}, |||_*), \quad x \mapsto J(x), \quad J(x)(f) := f(x) \forall f \in X^*
\]
is an embedding that is isometric, i.e., \( ||J(x)||_* = ||x|| \) for all \( x \in X \). Often the notation \( x^{**} := J(x) \) is used.

**Proof.** \( J \) is clearly linear. The estimate

\[
||J(x)||_* = \sup \{ ||J(x)|| \mid f \in X^*, ||f||_* = 1 \} = \sup \{ ||f(x)|| \mid f \in X^*, ||f||_* = 1 \} \leq ||f||_* ||x|| = ||x||.
\]
implies that \( J \) is bounded. Corollary 2.53 implies \( ||J(x)||_* = ||x|| \). Moreover, \( J \) is injective: Let us assume that \( J(x) = J(y) \) for \( x, y \in X \) with \( x \neq y \). This means \( f(x) = f(y) \) for all \( f \in X^* \), contradicting Corollary 2.52. \( \square \)

To the weak and weak* convergence, there are topologies associated, called the **weak topology** and the **weak* topology**.
Remark 3.7. 1) The weak topology is the smallest topology on $X$ such that all elements of $X^*$ are continuous. It is generated by the subbase
\[ \{(x^*)^{-1}(U) \mid x^* \in X^*, \ U \subseteq \mathbb{K} \text{ open}\}. \]
2) The weak* topology is the smallest topology on $X^*$ such that all elements of $J(X) \subseteq (X^*)^*$ are continuous on $X^*$. It is generated by the subbase
\[ \{(J(x))^{-1}(U) \mid x \in X, \ U \subseteq \mathbb{K} \text{ open}\}. \]

A natural question is if $J$ is also surjective. The answer is ‘generally not, but sometimes well’. More precisely:

Definition 3.8. A Banach space $X$ is reflexive if $J : (X, \| \cdot \|) \to (X^{**}, \| \cdot \|_{**})$ is surjective. In the literature, one usually finds the short notation $X = X^{**}$ instead of $J(X) = X^{**}$.

Many important spaces are reflexive:

Example 3.9. 1) All finite dimensional spaces are reflexive.
2) All Hilbert spaces are reflexive.
3) The Lebesgue spaces $L^p$ for $1 < p < \infty$ are reflexive with $(L^p)^* = L^q$ for $\frac{1}{p} + \frac{1}{q} = 1$.
4) The Lebesgue spaces $L^1$ and $L^\infty$ are reflexive if and only if they are finite dimensional.

Being reflexive is nicely ‘compatible’ with many important notions:

Proposition 3.10. 1) In reflexive spaces, the weak and the weak* convergence of sequences in the dual space are the same.
2) Closed subspaces of reflexive spaces are reflexive.
3) Let the spaces $X$ and $Y$ be isomorphic. Then $X$ reflexive $\iff$ $Y$ reflexive.
4) A space is reflexive if and only if its dual space is reflexive.

Proof. Cf. for instance [Alt]. \hfill \Box

2. Weak and weak* sequentially compactness of unit balls

We already know from previous courses that the unit ball of a metric or normed space is compact if and only if the space is finite dimensional.

The question now is if an infinite dimensional unit ball may be compact in some sense in the weak and/or weak* topology.

Definition 3.11. 1) A set $M$ is called weakly sequentially compact if every sequence in $M$ admits a weakly convergent subsequence.
2) A set $M$ is called weakly* sequentially compact if every sequence in $M$ admits a weakly* convergent subsequence.
The following theorem answers the question what kind of compactness we can get without additional assumptions.

**Theorem 3.12 (Banach-Alaoglu).** Let \((X, \| \|)\) be a Banach space over \(\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}\) and \((f_n)_{n \in \mathbb{N}} \in (X^*, \| \|_*)\) bounded. Then \((f_n)_{n \in \mathbb{N}}\) has a weakly* convergent subsequence whose weak* limit \(f \in X^*\) satisfies \(\|f\|_* \leq \sup_{n \in \mathbb{N}} \|f_n\|_*\). In particular, the closed unit ball in \((X^*, \| \|_*)\) is weakly* sequentially compact.

In 1932, Stefan Banach proved this statement under the assumption that \(X\) is separable. In 1940, the Greek-Canadian-US mathematician Leonidas (Leon) Alaoglu (1914 – 1981) proved the general case.

**Proof.** We only proof the statement for separable spaces and refer for the general case, which involves the axiom of choice, to the literature.

Let \((X, \| \|)\) be a separable Banach space and \(S = \{x_1, x_2, x_3, \ldots\}\) dense in \(X\). Let \((f_n)_{n \in \mathbb{N}} \in (X^*, \| \|_*)\) be bounded, i.e., there exists \(C > 0\) such that \(\|f_n\|_* \leq C\) for all \(n \in \mathbb{N}\). We have to show that there exists \(f \in X^*\) and a subsequence \((f_{n_j})_{j \in \mathbb{N}}\) with \(\lim_{j \to \infty} f_{n_j}(x) = f(x)\) for all \(x \in X\).

**Step 1:** We look for a subsequence that converges on \(S\). Consider \(x_1 \in S\) and notice that \(\|f_n(x_1)\| \leq C \|x_1\|\) is a bounded sequence in \(\mathbb{K}\). By the theorem of Bolzano-Weierstraß, there exists a converging subsequence \((f_{j_1})_{j_1 \in J_1} \subseteq \mathbb{N}\) and we set

\[
\lim_{j_1 \to \infty} f_{j_1}(x_1) =: f(x_1).
\]

We observe that \(\|f_{n_j}(x_2)\| \leq C \|x_2\|\) is bounded for all \(j \in J_1\). Thus, by the theorem of Bolzano-Weierstraß, there exists a converging subsequence \((f_j)_{j_2 \in J_2} \subseteq J_1\) and we set

\[
\lim_{j_2 \to \infty} f_{j_2}(x_2) =: f(x_2).
\]

Note that we still have \(\lim_{j_2 \to \infty} f_{j_2}(x_1) = f(x_1)\).

By iterating this procedure, we obtain a subsequence \((f_j)_{j \in I_k}\) that converges on \(x_1, \ldots, x_k\) to \(f(x_1), \ldots, f(x_k)\). For \(k \to \infty\), we get an infinite subset \(I_\infty \subseteq \mathbb{N}\) and a subsequence \((f_j)_{j \in I_\infty}\) converging on all of \(S\) to a function \(f : S \to \mathbb{K}\).

**Step 2:** We now show that \(f : S \to \mathbb{K}\) is \(C\)-Lipschitz: For \(x_r, x_s \in S\), we estimate

\[
|f(x_r) - f(x_s)| = \lim_{j_\infty \to \infty} |f_{j_\infty}(x_r) - f_{j_\infty}(x_s)| \leq \sup_{j_\in I_\infty} \|f_{j_\infty}\|_* \|x_r - x_s\| \\
\leq C \|x_r - x_s\|.
\]
Since $f$ is Lipschitz on $S$, we can extend $f$ by continuity to $\overline{S} = X$. We denote the extension by $f : X \to \mathbb{K}$.

Step 3: Now we show that $(f_j)_{j \in J_\infty}$ converges weakly* to $f$ on $X$. Let $\varepsilon > 0$ and consider an arbitrary $x \in X$. Since $S$ is dense in $X$ there is $x_k$ in $S$ such that $\|x - x_k\| < \varepsilon$. Moreover, there is $N \in J_\infty$ such that $|f_j(x_k) - f(x_k)| < C\varepsilon$ for all $j \geq N$. Then we get for all $j \in J_\infty$ with $j \geq N$

$$
\left| f_j(x) - f(x) \right| \leq \left| f_j(x) - f(x_k) \right| + \left| f_j(x_k) - f(x_k) \right| + \left| f(x_k) - f(x) \right|
\leq C \|x - x_k\| + C\varepsilon + C \|x_k - x\|
< 3C\varepsilon,
$$

meaning

$$
\lim_{j \to \infty, j \in J_\infty} f_j(x) = f(x).
$$

Since this works for all $x \in X$, the subsequence $(f_j)_{j \in J_\infty}$ converges weakly* to $f$ on $X$. Since $f$ is the pointwise limit of a uniformly bounded family of continuous linear operators, Theorem 2.17 (Banach-Steinhaus) implies $f \in X^*$ with $\|f\|_* \leq \sup_{j \in J_\infty} \|f_j\|_* \leq C$. \hfill $\square$

As an exercise, one can show that the closed unit ball in a Hilbert space is weakly sequentially compact. But, in fact, one can do better. Since we only proved Theorem 3.12 (Banach-Alaoglu) for separable Banach spaces we feel somewhat obliged to use in the following the separable version. This means that we will need the following statement, before we can study weak or weak* sequentially compactness of unit balls in reflexive spaces.

**Lemma 3.13.** $X^*$ separable $\Rightarrow X$ separable.

**Proof.** Let $\{f_n \in X^* \mid n \in \mathbb{N}\}$ be dense in $X^*$. For all $n \in \mathbb{N}$, choose $x_n \in X$ such that $|f_n(x_n)| \geq \frac{1}{2} \|f_n\|_*$ and $\|x_n\| = 1$. Now set

$$
Y := \text{Span}_\mathbb{K}\{x_n \mid n \in \mathbb{N}\} \subseteq X.
$$

Using the span over $\mathbb{Q} \subset \mathbb{R}$ resp. over $\mathbb{Q}^2 \subset \mathbb{C}$, we see that $Y$ is separable. Assume that $Y \neq X$. For all $f \in X^*$ with $f|_Y = 0$, we estimate

$$
\|f - f_n\|_* \geq |(f - f_n)(x)| = |f_n(x)| \geq \frac{1}{2} \|f_n\|_* \geq \frac{1}{2} (\|f\|_* - \|f_n - f\|_*)
$$

which implies $3 \|f - f_n\|_* \geq \frac{1}{2} \|f\|_*$. This in turn implies

$$
3 \inf_{n \in \mathbb{N}} \|f - f_n\|_* \geq \frac{1}{2} \|f\|_*.
$$

Since $(f_n)_{n \in \mathbb{N}}$ is dense in $X^*$ we have $\inf_{n \in \mathbb{N}} \|f - f_n\|_* = 0$ and thus $\|f\|_* = 0$ which implies $f \equiv 0$ for all $f \in X^*$ with $f|_Y = 0$. But this contradicts Corollary 2.54, so we must have $Y = X$, i.e., $X$ is separable. \hfill $\square$

Now we are ready to proof the following.
3. WEAK AND WEAK* CONVERGENCE AND COMPACTNESS

Theorem 3.14. Let \((X, \|\|)\) be a reflexive Banach space over \(K \in \{\mathbb{R}, \mathbb{C}\}\). Then every bounded sequence in \(X\) has a weakly convergent subsequence. In particular, the closed unit ball of \(X\) is weakly sequentially compact.

Proof. Let \((x_n)_{n \in \mathbb{N}} \in X\) be bounded, i.e., there exists \(C > 0\) such that \(\|x_n\| \leq C\) for all \(n \in \mathbb{N}\). Define

\[
\text{Span}_{\mathbb{K}} \{x_n \mid n \in \mathbb{N}\} =: Y
\]

Since \(X\) is reflexive and \(Y \subseteq X\) a closed subspace, \(Y\) is reflexive by Proposition 3.10. Using the span over \(\mathbb{Q}\) \(\subset \mathbb{R}\) resp. over \(\mathbb{Q}^2 \subseteq \mathbb{C}\), we see that \(Y\) is separable. By Proposition 3.10, \(Y^{**}\) is therefore reflexive, too. Since \((Y^*)^* = Y^{**}\), Lemma 3.13 implies \(Y^*\) to be separable. We use the notation \(J(x_n) = x_n^{**}\) and find

\[
\|x_n^{**}\|_{**} = \|J(x_n)\|_{**} = \|x_n\| \leq C
\]

such that, by Theorem 3.12 (Banach-Alaoglu), which we proved for the separable case, there exists \(x^{**} \in X^{**}\) and a subsequence \((x_{n_j}^{**})_{j \in \mathbb{N}}\) that converges weakly* to \(x^{**}\). We set \(x := J^{-1}(x^{**}) \in Y\) and obtain

\[
\lim_{j \to \infty} f(x_{n_j}) = \lim_{j \to \infty} x_{n_j}^{**}(f) = x^{**}(f) = f(x)
\]

for all \(f \in Y^*\). Moreover, since \(f \in X^*\) yields \(f|_Y \in Y^*\) and since \(x, (x_{n_j})_{j \in \mathbb{N}} \in Y\), we conclude \(\lim_{j \to \infty} f(x_{n_j}) = f(x)\) for all \(f \in X^*\), meaning \((x_{n_j})_{j \in \mathbb{N}}\) converges weakly to \(x\). \(\square\)

3. Weak convergence of probability measures

We are now able to show a link between functional analysis and measure theory by identifying weak* convergence on the dual space of continuous functions with weak convergence of probability measures. We start with recalling some notation and facts from measure theory.

Let \(S \subseteq \mathbb{R}^n\) be bounded and assume \(S\) to be in addition either open or closed. Denote by \(\mathcal{B}(S)\) the \(\sigma\)-algebra of Borel sets which is generated by the open subsets of \(S\). With an additive function \(\mu : \mathcal{B}(S) \to \mathbb{K}^m\), we can associate the variation measure \(|\mu| : \mathcal{B}(S) \to [0, \infty]\) given by

\[
|\mu|(E) := \sup \left\{ \sum_{i=1}^k |\mu(E_i)| \mid k \in \mathbb{N}, E_1, \ldots, E_k \in \mathcal{B}(S) \text{ pairwise disjoint, } E_1, \ldots, E_k \subseteq E \right\}
\]

which is an additive function. The total variation of \(\mu\) is defined by

\[
||\mu||_{\text{var}} := |\mu(S)|
\]

and we consider the space

\[
ca(S, \mathbb{K}^m) := \{\mu : \mathcal{B}(S) \to \mathbb{K}^m \mid \mu \text{ } \sigma\text{-additive, } ||\mu||_{\text{var}} < \infty\}
\]
where \( ca \) stands for *countably additive*. We note that
\[
(ca(S, \mathbb{K}^m), \|\|_{\text{var}})
\]
is a Banach space (cf. [Alt]). A measure \( \mu \in ca(S, \mathbb{K}^m) \) is said to be *regular* if we have for all \( E \in \mathcal{B}(S) \)
\[
0 = \inf \{ |\mu|(U \setminus G) | G \subset E \subset U, G \text{ closed in } S, U \text{ open in } S \}.
\]
We define the space
\[
rca(S, \mathbb{K}^m) := \{ \mu \in ca(S, \mathbb{K}^m) | \mu \text{ regular} \}
\]
and note that
\[
(rca(S, \mathbb{K}^m), \|\|_{\text{var}})
\]
is a Banach space (cf. [Alt]).

**Theorem 3.15 (Riesz-Radon).** Let \( S \subset \mathbb{R}^n \) be compact. Then
\[
\alpha : rca(S, \mathbb{K}) \to (C^0(S, \mathbb{K}))^*, \quad \mu \mapsto \alpha(\mu)
\]
defined by
\[
\alpha(\mu)(f) := \int_S f \, d\mu \quad \forall \ f \in C^0(S, \mathbb{K})
\]
is a linear, continuous, isometric isomorphism.

**Proof.** Cf. for instance [Alt]. \( \square \)

We say that a sequence \( (\mu_n)_{n \in \mathbb{N}} \in rca(S, \mathbb{K}) \) **converges weakly** to \( \mu \in rca(S, \mathbb{K}) \) if
\[
\lim_{n \to \infty} \int_S f \, d\mu_n = \int_S f \, d\mu \quad \forall \ f \in C^0(S, \mathbb{K}).
\]
A measure \( \mu \in rca(S, \mathbb{K}) \) is a *probability measure* if \( \mu \) is positive and \( \mu(S) = 1 \).

**Corollary 3.16.** Let \( S \subset \mathbb{R}^n \) be compact. Then the subset of probability measure in \( rca(S, \mathbb{K}) \) is weakly sequentially compact, i.e., each sequence of probability measures \( (\mu_n)_{n \in \mathbb{N}} \in rca(S, \mathbb{K}) \) has a subsequence \( (\mu_{n_j})_{j \in \mathbb{N}} \) that converges weakly to some probability measure \( \mu \in rca(S, \mathbb{K}) \).

**Proof.** Keep in mind that the polynomials are dense in \( C^0(S, \mathbb{K}) \) for compact \( S \subset \mathbb{R}^n \), i.e., \( C^0(S, \mathbb{K}) \) is separable. Using 3.15 (Riesz-Radon), we can identify weak convergence of probability measures in \( rca(S, \mathbb{K}) \) with weak* convergence in \( (C^0(S, \mathbb{K}))^* \). By Theorem 3.12 (Banach-Alaoglu), we get weak* sequentially compactness of the unit ball of \( (C^0(S, \mathbb{K}))^* \). We leave the details to the reader. \( \square \)

In probability theory, Corollary 3.16 is known as **Prohorov’s theorem**.
Fredholm theory

Fredholm operators naturally show up in spectral theory of compact operators and in applications of functional analysis to topology, geometry, and partial differential equations. A good reference for operator theory in general and Fredholm theory in particular is [Simon, Section 3.15]. Geometric applications are given in [McDuff & Salamon, Appendix A].

1. Definitions and examples

Let us start with the following important class of operators.

Definition 4.1. Let $X, Y$ be Banach spaces. $T \in \mathcal{B}(X, Y)$ is compact if, for all bounded sequences $(x_n)_{n \in \mathbb{N}} \in X$, the sequence $(T(x_n))_{n \in \mathbb{N}} \in Y$ has a strongly convergent subsequence. We denote the space of compact operators from $X$ to $Y$ by $\mathcal{K}(X, Y)$.

Compact operators have many convenient properties, among others nice spectral properties concerning eigenvalues etc. For details we refer to previous courses on functional analysis in the bachelor program and/or any introductory books on functional analysis like [Alt], [Bressan], [Haase].

The following important class of operators is named after the Swedish mathematician Erik Ivar Fredholm (1866 – 1927).

Definition 4.2. Let $X, Y$ be Banach spaces.
1) $F \in \mathcal{B}(X, Y)$ is a Fredholm operator if
   - $\dim(\ker F) < \infty$.
   - $\text{range}(F)$ is closed in $Y$.
   - $\dim(\text{coker } F) := \dim(Y/\text{range}(F)) < \infty$.
2) The Fredholm index of a Fredholm operator $F$ is given by
   $$\text{Ind}(F) := \dim(\ker F) - \dim(\text{coker } F) < \infty.$$  

We denote the space of Fredholm operators from $X$ to $Y$ by $\mathcal{F}(X, Y)$.

When working with dimensions, one has to pay attention to the underlying field, i.e., if one is working over the complex or real numbers. The Fredholm index transform via $\text{Ind}_{\mathbb{R}} = 2 \text{Ind}_{\mathbb{C}}$.

Example 4.3. The identity is a Fredholm operator.
In perturbation theory, operators of the form $(\text{Id} - K)$ with $K \in \mathcal{K}(X, Y)$ are considered as ‘compact perturbations’ of the identity. Such operators are of importance in the spectral theory of compact operators since the associated eigenvalue equation can be seen as

$$K - \lambda \text{Id} = 0 \iff \text{Id} - \frac{1}{\lambda} K = 0 \quad \text{for } \lambda \in \mathbb{C}^\neq 0.$$ 

**Theorem 4.4.** Let $X$ be Banach and $K \in \mathcal{K}(X, X)$. Then $(\text{Id} - K)$ is a Fredholm operator with Fredholm index $\text{Ind}(\text{Id} - K) = 0$.

**Sketch of proof.** Since the scope of this course does not include spectral theory of compact operators, we only sketch the line of thought of this quite lengthy proof and refer for details to the literature, see for instance [Bressan] (for $X$ real Hilbert space; shorter proof) or [Alt] (for $X$ Banach space). Usually the claim is shown by proving the following steps successively.

1. $\dim(\ker(\text{Id} - K)) < \infty$.
2. $\text{range}(\text{Id} - K)$ is closed.
3. $\ker(\text{Id} - K) = \{0\} \Rightarrow \text{range}(\text{Id} - K) = X$.
4. $\text{codim}(\text{range}(\text{Id} - K)) \leq \dim(\ker(\text{Id} - K))$.
5. $\dim(\ker(\text{Id} - K)) \leq \text{codim}(\text{range}(\text{Id} - K))$.

\[\square\]

2. Adjoint operator on Banach spaces

Let $(X, \langle \ , \ \rangle_X)$ and $(Y, \langle \ , \ \rangle_Y)$ be Hilbert spaces and $T \in \mathcal{B}(X, Y)$. Then the **Hilbert adjoint operator** $T^* \in \mathcal{B}(Y, X)$ of $T$ is defined via the equation

$$\langle x, T^*(y) \rangle_X = \langle T(x), y \rangle_Y \quad \forall \ x \in X, \forall \ y \in Y.$$ 

This prompts the question wether we can define some analogue of the Hilbert adjoint on Banach spaces. In particular, this requires finding some substitute for the scalar products that do not exist in the Banach setting.

**Definition 4.5.** Let $X$ be a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $X^*$ its dual. We define the **dual paring** of these two spaces via

$$\langle \ , \ \rangle : X \times X^* \to \mathbb{K}, \quad \langle x, x^* \rangle := x^*(x).$$

Now we are ready to extend the notion of Hilbert adjoint to Banach spaces.

**Definition and Lemma 4.6.** Let $X$, $Y$ be Banach spaces with duals $X^*$, $Y^*$. Then

$$* : \mathcal{B}(X, Y) \to \mathcal{B}(Y^*, X^*), \quad T \mapsto T^*$$

given by

$$\langle x, T^*(y^*) \rangle = \langle T(x), y^* \rangle \quad \forall \ x \in X, \forall \ y^* \in Y^*$$
is an isometric imbedding, i.e., coinciding operator norms $\|T\| = \|T^*\|$. The operator $T^*$ is called the Banach adjoint operator of $T$, briefly the adjoint operator.

Proof. The map $T \mapsto T^*$ is surely linear. We show it being an isometry. Denote the norm on $X$ by $\|\|_X$ etc. We first show $\|T^*\| \leq \|T\|$ for which we estimate

$$|\langle x, T^*(y^*) \rangle| = |\langle T(x), y^* \rangle| \leq \|T\| \|x\|_X \|y^*\|_Y,$$

which implies

$$\left\| \frac{x}{\|x\|_X}, T^*(y^*) \right\| \leq \|T\| \|y^*\|_Y.$$

and, by taking the supremum over $x \in X$, this implies $\|T^*(y^*)\| \leq \|T\| \|y^*\|_Y$ and therefore $\|T^*\| \leq \|T\|$.

To show $\|T^*\| \geq \|T\|$, we estimate for all $x \in X$ with $\|x\|_X \leq 1$ and all $y^* \in Y^*$ with $\|y^*\|_{Y^*} \leq 1$

$$\|T^*\| \geq \|T^*(y^*)\| \geq |\langle T^*(y^*)(x) \rangle| = |\langle x, T^*(y^*) \rangle| = |\langle T(x), y^* \rangle|.$$

Corollary 2.53 implies the existence of $y^* \in Y^*$ with $\|y^*\|_{Y^*} = 1$ and $(T^*(y^*))(x) = \langle x, T^*(y^*) \rangle = \langle T(x), y^* \rangle = \|T(x)\|_Y$. This yields $\|T^*\| \geq \|T(x)\|_Y$ for all $x \in X$ with $\|x\|_X \leq 1$, hence in particular $\|T^*\| \geq \|T\|$. \hfill $\square$

The Banach adjoint generalizes indeed the Hilbert adjoint as we will see now. Recall first the following essential result from Hilbert theory named after the Hungarian mathematician Frigyes Riesz (1880 – 1956).

**Theorem 4.7 (Riesz).** Let $(X, \langle \cdot, \cdot \rangle_X)$ be a Hilbert space. Then

1) $\forall y \in X$: the map $x \mapsto \langle x, y \rangle$ is a continuous linear functional.

2) $\forall T \in \mathcal{B}(X, X) : \exists! x_T \in X : T(x) = \langle x, x_T \rangle.$

3) The map $R_Y : X \to X^*, x \mapsto R_Y(x)$ with $R_Y(x)(z) := \langle z, x \rangle_X$ for all $z \in X$ is an isometric isomorphism.

Proof. See for example [Alt] or [Bressan]. \hfill $\square$

We note

**Remark 4.8.** Consider the Hilbert spaces $(X, \langle \cdot, \cdot \rangle_X)$ and $(Y, \langle \cdot, \cdot \rangle_Y)$ and $T \in \mathcal{B}(X, Y)$ with Hilbert adjoint $T^* \in \mathcal{B}(Y, X)$. Then

$$T^* = R_Y \circ T^* \circ (R_Y)^{-1} \in \mathcal{B}(Y^*, X^*).$$

Passing to the Banach adjoint is compatibel with taking the inverse and being compact:

**Lemma 4.9.** Let $X, Y$ be Banach and $T \in \mathcal{B}(X, Y)$. Then

1) $\exists T^{-1} \in \mathcal{B}(Y, X) \iff \exists (T^*)^{-1} \in \mathcal{B}(X^*, Y^*)$.

2) $T \in \mathcal{K}(X, Y) \iff T^* \in \mathcal{K}(Y^*, X^*)$. 

4. FREDHOLM THEORY

Proof. Left as an exercise to the reader or see [Alt] or [Bressan].

Using the dual pairing we can generalize the notion of orthogonality from Hilbert spaces to Banach spaces:

**Definition 4.10.** Let $X$ be a Banach space and $Z \subseteq X$ a subspace. Then

$$\text{Ann}(Z) := \{ x^* \in X^* | \langle z, x^* \rangle = 0 \ \forall \ z \in Z \}$$

is called the annihilator of $Z$.

In the literature, the annihilator of a subspace $Z$ is sometimes also denoted by $Z^0$ or $Z^\perp$. It consists of all bounded linear functionals that vanish on $Z$.

**Remark 4.11.** In Hilbert spaces, the annihilator $\text{Ann}(Z)$ of a subspace $Z$ is isomorphic to the orthogonal complement $Z^\perp$ of $Z$ by Theorem 4.7 (Riesz).

Moreover, the annihilator has the following important geometric meaning.

**Lemma 4.12.** Let $X, Y$ be Banach spaces and $T \in \mathcal{B}(X, Y)$. Moreover, let $Z \subseteq X$ be a closed subspace with $\text{codim}(Z) < \infty$. Then

1) $\ker(T^*) = \text{Ann}(\text{range}(T))$.
2) $\dim(\text{Ann}(Z)) = \text{codim}(Z)$.

**Proof.** 1) We find

$$y^* \in \ker(T^*) \iff 0 = \langle x, T^*(y^*) \rangle = \langle T(x), y^* \rangle \iff y^* \in \text{Ann}(\text{range}(T)).$$

2) Let $x_1, \ldots, x_n \in X$ linear independent with $X = Z \oplus \text{Span}\{x_1, \ldots, x_n\}$ and consider the functionals $x_1^*, \ldots, x_n^* \in X^*$ satisfying $\langle x_i, x_j^* \rangle = \delta_{ij}$ (Kronecker symbol). Then the functionals $x_1^*, \ldots, x_n^*$ are linearly independent and lie in $\text{Ann}(Z)$ implying $n \leq \dim(\text{Ann}(Z))$. On the other hand, given some $f \in \text{Ann}(Z)$ and some $x = z + \sum_{i=1}^n r_i x_i$ with $z \in Z$ and scalars $r_1, \ldots, r_n$, we compute

$$\langle x, f \rangle = \sum_{i=1}^n r_i \langle x_i, f \rangle = \sum_{i,j=1}^n r_i \langle x_j, f \rangle x_i^* = \sum_{j=1}^n \langle x_j, f \rangle x_j^*$$

$$= \left( x, \sum_{j=1}^n \langle x_j, f \rangle x_j^* \right).$$

Thus $f$ can be written as linear combination of the $x_1^*, \ldots, x_n^*$ such that we obtain $n \geq \dim(\text{Ann}(Z))$. Hence $\text{codim}(Z) = n = \dim(\text{Ann}(Z))$. 

□
3. Fredholm alternative

Now we interpret equations induced by Fredholm operators under the point of view of ‘dimension of the solution space’ and ‘dimension of solvability constraints’. Roughly, the first is given by the kernel and the latter by the cokernel of the Fredholm operator. In fact, there are precisely two possible scenarios. For that reason, one speaks of the so-called Fredholm alternative:

**Theorem 4.13 (Fredholm alternative).** Let $X$ be Banach and $K \in \mathcal{K}(X, X)$. Then

1) If $\ker(\text{Id} - K) = \{0\}$, then the homogeneous equation $(\text{Id} - K)(u) = 0$ has only the trivial solution $u = 0$. In this case, $(\text{Id} - K)$ is invertible and, for each $v \in X$, there exists exactly one solution $u_v := (\text{Id} - K)^{-1}(v) \in X$ solving the equation

$$(\text{Id} - K)(u) = v.$$ 

2) If $\ker(\text{Id} - K) \neq \{0\}$, then the homogeneous equation $(\text{Id} - K)(u) = 0$ has nontrivial solutions. In this case, the equation

$$(\text{Id} - K)(u) = v$$

has solutions $u$ for a given $v \in X$ if and only if $v$ satisfies the following constraints:

$$\langle v, \varphi \rangle = 0 \text{ for all } \varphi \in \ker(\text{Id} - K^*) = \text{Ann}(\text{range}(\text{Id} - K)).$$

In particular, the dimensions of the space of ‘solvability constraints’ $\varphi$ on $v \in X$ coincides with the dimension of the solution space of the homogeneous equation $(\text{Id} - K)(u) = 0$.

**Proof.** Let $K \in \mathcal{K}(X, X)$. By Theorem 4.4, the operator $(\text{Id} - K)$ is Fredholm with $\dim(\ker(\text{Id} - K)) = \dim(\text{coker}(\text{Id} - K))$. Now consider:

1) If $\ker(\text{Id} - K) = \{0\}$ then $F := (\text{Id} - K)$ is injective. Moreover,

$$0 = \dim(\ker(\text{Id} - K)) \overset{\text{4.4}}{=} \dim(\text{coker}(\text{Id} - K))$$

implies $\text{range}(\text{Id} - K) = X$, i.e., $(\text{Id} - K)$ is also surjective. By Theorem 2.29 (Inverse mapping), the inverse operator exists and is continuous. This means that, given $v \in X$, precisely $u_v := (\text{Id} - K)^{-1}(v) \in X$ solves the equation $(\text{Id} - K)(u) = v$. 


2) We observe:

\[(\text{Id} - K)(u) = v\] has a solution \(u \in X\) for a given \(v \in X\).

\[\iff v \in \text{range}((\text{Id} - K)).\]

\[\iff \langle v, \varphi \rangle = 0\] for all \(\varphi \in \text{Ann}(\text{range}(\text{Id} - K)).\)

\[\iff 4.12 \langle v, \varphi \rangle = 0\] for all \(\varphi \in \ker((\text{Id} - K)^*) = \ker(\text{Id} - K^*).\)

Moreover, since \((\text{Id} - K)\) is Fredholm, \(\dim(\ker(\text{Id} - K))\) is finite dimensional and \(\text{range}(\text{Id} - K) \subseteq X\) is closed and has finite codimension. We conclude

\[
\dim(\ker(\text{Id} - K)) = \dim(\ker(\text{Id} - K^*)),
\]

\[\text{dim}(\text{ker}(\text{Id} - K)) = \text{dim}(\text{coker}(\text{Id} - K)) = \text{codim}(\text{range}(\text{Id} - K)) = \text{dim}(\text{Ann}(\text{range}(\text{Id} - K))).
\]

Furthermore, we have in particular

\[0 < \dim(\ker(\text{Id} - K)) = \dim(\ker(\text{Id} - K^*)) < \infty.\]

\[\square\]

4. Persistence under perturbations

So far, we only had compact perturbations of the identity as explicit examples for Fredholm operators. In this paragraph, we are interested in more general Fredholm operators. This raises the question how to prove that an operator is Fredholm. A major help is the following statement.

**Lemma 4.14 (Semi-Fredholm estimate).** Let \((X, \|\cdot\|_X), (Y, \|\cdot\|_Y)\) and \((Z, \|\cdot\|_Z)\) be Banach spaces, \(D \in \mathcal{B}(X,Y)\) and \(K \in \mathcal{K}(X,Z)\). Assume that there exists \(c > 0\) such that

\[
\|x\|_X \leq c(\|D(x)\|_Y + \|K(x)\|_Z) \quad \forall x \in X.
\]

Then \(\text{range}(D)\) is closed and \(\dim(\ker(D)) < \infty.\)

**Proof.** We first show \(\dim(\ker(D)) < \infty\). This is equivalent to showing that the unit ball of \(\ker(D)\) is compact. In metric spaces, this is equivalent to showing that the unit ball of \(\ker(D)\) is sequentially compact in the topology induced by \(\|\cdot\|_X\). To show this, let \((x_n)_{n \in \mathbb{N}} \in \ker(D)\) with \(\|x_n\|_X \leq 1\). Since \((x_n)_{n \in \mathbb{N}}\) is bounded and \(K\) compact, the sequence \((K(x_n))_{n \in \mathbb{N}}\) has a convergent subsequence \((K(x_{n_j}))_{j \in \mathbb{N}}\) which is in particular a Cauchy sequence. Since \(D(x_{n_j}) = 0\) for all \(j \in \mathbb{N}\), we conclude from

\[
\|x_{n_j} - x_m\|_X \leq c \left(\|D(x_{n_j}) - D(x_m)\|_Y + \|K(x_{n_j}) - K(x_m)\|_Z\right)
\]

\[= c \|K(x_{n_j}) - K(x_m)\|_Z\]
that \((x_{n_j})_{j \in \mathbb{N}}\) is a Cauchy sequence as well. Since \(X\) is Banach, there exists \(x \in X\) with \(\lim_{j \to \infty} x_{n_j} = x\). Since \(\ker(D)\) is closed, we have \(x \in \ker(D)\).

Now we show that range\((D)\) is closed. Let \((x_n)_{n \in \mathbb{N}} \in X\) with \(\lim_{n \to \infty} D(x_n) =: y \in Y\). We have to show \(y \in \text{range}(D)\). The sequence \((x_n)_{n \in \mathbb{N}}\) is either bounded or unbounded in the norm topology.

Let us first study the case \((x_n)_{n \in \mathbb{N}}\) bounded. Then \((K(x_n))_{n \in \mathbb{N}}\) has a convergent subsequence \((x_{n_j})_{j \in \mathbb{N}}\) which is a Cauchy sequence due to

\[
\|x_{n_j} - x_{n_k}\|_X \leq c \left( \|D(x_{n_j}) - D(x_{n_k})\|_Y + \|K(x_{n_j}) - K(x_{n_k})\|_Z \right).
\]

Since \(X\) is Banach, there exists \(\lim_{j \to \infty} x_{n_j} =: x\) and we get by continuity \(y = \lim_{j \to \infty} D(x_{n_j}) = D(x)\).

Now we show that such an \((x_n)_{n \in \mathbb{N}}\) cannot be unbounded. Assume that \((x_n)_{n \in \mathbb{N}}\) is unbounded. Then there exists a subsequence \((x_{n_j})_{j \in \mathbb{N}}\) with \(\lim_{j \to \infty} \|x_{n_j}\| = \infty\). Since \(\ker(D)\) is finite dimensional, \(X\) splits into \(X = \ker(D) \oplus X_0\) with some space \(X_0\). Since we are interested in range\((D)\), we can assume w.l.o.g. that \((x_{n_j})_{j \in \mathbb{N}} \in X_0\). Then \(u_j := \frac{x_{n_j}}{\|x_{n_j}\|_X} \in X_0\) has \(\|u_j\|_X = 1\) and satisfies due to (4.15)

\[
\|u_j\|_X \leq c \left( \|D(u_j)\|_Y + \|K(u_j)\|_Z \right) = c \left( \|D(x_{n_j})\|_Y \|x_{n_j}\|_X + \|K(u_j)\|_Z \right).
\]

The limits \(\lim_{j \to \infty} D(x_{n_j}) = y\) and \(\lim_{j \to \infty} \|x_{n_j}\| = \infty\) imply that \(\lim_{j \to \infty} \|D(x_{n_j})\|_Y \|x_{n_j}\|_X = 0\). Since \((u_j)_{j \in \mathbb{N}}\) is bounded \((K(u_j))_{j \in \mathbb{N}}\) has a converging subsequence \((K(u_{j_k}))_{k \in \mathbb{N}}\) and we conclude as above that \((u_{j_k})_{k \in \mathbb{N}}\) is a Cauchy sequence in \(X_0\) with \(\lim_{k \to \infty} u_{j_k} =: u\). It satisfies \(\|u\|_X = 1\) and \(D(u) = 0\). Thus we have \(u \in X_0 \cap \ker(D) = \{0\}\) implying \(u = 0\) in contradiction to \(\|u\|_X = 1\).

□

We want to show that being Fredholm persists under perturbation with operators having sufficiently small operator norm as well as under perturbations with arbitrary compact operators. We start with

**Lemma 4.16.** Let \(X, Y\) be Banach spaces, \(D \in \mathcal{B}(X, Y)\) with closed \(\text{range}(D)\) and \(\dim(\ker(D)) < \infty\), and \(K \in \mathcal{K}(X, Y)\). Then

1) \(\text{range}(D + K)\) is closed and \(\dim(\ker(D + K)) < \infty\).

2) There is \(\varepsilon > 0\) such that, for all \(P \in \mathcal{B}(X, Y)\) with \(\|P\| < \varepsilon\), the subspace \(\text{range}(D + P)\) is closed and \(\dim(\ker(D + P)) < \infty\).

3) One may replace everywhere in this lemma \(\ker\) by \(\text{coker}\) and the statements continue to hold true.

**Proof.** We want to use Lemma 4.14 (Semi-Fredholm estimate), but to do so, we need an inequality comparable to (4.15).
Let \( n := \dim(\ker(D)) \) and extend the identification \( \ker(D) \cong \mathbb{R}^n \) by means of Theorem 2.51 (Hahn-Banach) to an operator \( L \in \mathcal{B}(X, \mathbb{R}^n) \). Define

\[
T : X \to Y \oplus \mathbb{R}^n, \quad T(x) := (D(x), L(x))
\]

and consider the splitting \( X =: \bar{X} \oplus \ker(D) \). The operator \( T \) is injective since, assuming \( T(x) = T(y) \), we find for \( x = \bar{x} \oplus x_D, z = \bar{z} \oplus z_D \in \bar{X} \oplus \ker(D) \)

\[
(D(\bar{x}), L(x_D)) = T(x) = T(z) = (D(\bar{z}), L(z_D))
\]

implying \( x_D = z_D \) and \( \bar{x} = \bar{z} \) such that \( x = z \). Moreover, since \( \text{range}(D) \) is closed,

\[
\text{range}(T) = \{(D(\bar{x}), L(x_D)) \mid \bar{x} \oplus x_D \in \bar{X} \oplus \ker(D)\} \cong \text{range}(D) \times \mathbb{R}^n
\]

is closed w.r.t. the topology generated by the norm \( \|\| := \|\|_Y + \|\|_{\text{eucl}} \) on \( Y \oplus \mathbb{R}^n \). As closed subspace of the Banach space \( Y \oplus \mathbb{R}^n \), \( \text{range}(T) \) is also Banach. Theorem 2.29 (Inverse mapping) yields the existence of \( T^{-1} \in \mathcal{B}(\text{range}(T), X) \) which is in particular bounded, i.e., there is \( c > 0 \) such that

\[
\|T^{-1}(y)\|_X \leq c \|y\|.
\]

Written as \( y = T(x) = (D(\bar{x}), L(x_D)) \) we get

\[
\|x\|_X = \|T^{-1}(T(x))\|_X \leq c \|T(x)\| = c (\|D(\bar{x})\|_Y + \|L(x_D)\|_{\text{eucl}}).
\]

Since \( \dim(\text{range}(L)) < \infty \), the operator \( L \) is compact. Thus we have now Lemma 4.14 (Semi-Fredholm estimate) at our disposal.

Now let \( K \in \mathcal{K}(X, Y) \) and \( P \in \mathcal{B}(X, Y) \) and consider the operators

\[
F_1 := (D + K, K, L) : X \to Y \oplus Y \oplus \mathbb{R}^n, \\
F_2 := (D + P, P, L) : X \to Y \oplus Y \oplus \mathbb{R}^n.
\]

Assuming \( \|P\|_{op} \) small enough, one can deduce analogously as above the existence of \( c_1, c_2 > 0 \) such that

\[
\|x\|_X \leq c_1 (\|(D + K)(x)\|_Y + \|K(x)\|_Y + \|L(x)\|_{\text{eucl}}),
\]

\[
\|x\|_X \leq c_2 (\|(D + P)(x)\|_Y + \|P(x)\|_Y + \|L(x)\|_{\text{eucl}}).
\]

Using \( \|P(x)\|_Y \leq \|P\|_{op} \|x\|_X \), we can transform the second equation into

\[
\|x\|_X \leq \frac{c_2}{1 - c_2 \|P\|_{op}} (\|(D + P)(x)\|_Y + \|L(x)\|_{\text{eucl}}).
\]

Lemma 4.14 (Semi-Fredholm estimate) implies now that \( D + K \) and \( D + P \) have closed range and finite dimensional kernel. \( \square \)

We will see now that being Fredholm persists when passing to the adjoint operator. Moreover, the Fredholm index behaves additive under concatenation of Fredholm operators. Furthermore, Fredholm operators are intimately related to compact operators.

**Proposition 4.17.** Let \( X, Y, Z \) be Banach spaces and \( F \in \mathcal{B}(X, Y) \).
5. Inverse and Implicit Function Theorems and Sard’s Theorem

1) If \( F \in \mathcal{F}(X,Y) \) if and only if there exists \( T \in \mathcal{B}(Y,X) \) such that \( (F \circ T - \text{Id}) \) and \( (T \circ F - \text{Id}) \) are compact operators.

2) \( F_1 \in \mathcal{F}(X,Y) \) and \( F_2 \in \mathcal{B}(Y,Z) \) implies \( F_2 \circ F_1 \in \mathcal{B}(X,Z) \) and 
\[ \text{Ind}(F_2 \circ F_1) = \text{Ind}(F_1) + \text{Ind}(F_2). \]

3) \( F \in \mathcal{F}(X,Y) \iff F^* \in \mathcal{F}(Y^*,X^*) \). In particular 
\[ \text{Ind}(F^*) = -\text{Ind}(F). \]

**Proof.** Left to the reader as an exercise. \( \square \)

We finally show that being Fredholm is stable under small perturbations as well as under arbitrary compact perturbations.

**Theorem 4.18.** Let \( X, Y \) be Banach spaces and \( F \in \mathcal{F}(X,Y) \). Then

1) \( \forall K \in \mathcal{K}(X,Y) : (F + K) \in \mathcal{F}(X,Y) \) and \( \text{Ind}(F + K) = \text{Ind}(F) \).

2) \( \exists \varepsilon > 0 : \forall T \in \mathcal{B}(X,Y) \) with \( \|T\| < \varepsilon \) :
\[ (F + T) \in \mathcal{F}(X,Y) \quad \text{and} \quad \text{Ind}(F + T) = \text{Ind}(F). \]

**Proof.** See for example [McDuff & Salamon, Appendix A]. \( \square \)

Topologically, Theorem 4.18 means

**Corollary 4.19.** Let \( X, Y \) be Banach spaces. Then \( \mathcal{F}(X,Y) \) is open in \( \mathcal{B}(X,Y) \) w.r.t. to the topology generated by the operator norm. Moreover, the Fredholm index is constant on each connected component of \( \mathcal{F}(X,Y) \).

5. Inverse and Implicit Function Theorems and Sard’s Theorem

In this section, we discuss the importance of Fredholm theory for differential calculus between Banach spaces. We begin with

**Definition 4.20.** Let \( (X, \|\cdot\|_X) \) and \( (Y, \|\cdot\|_Y) \) be Banach spaces and \( U \subseteq X \) open. \( f : U \to Y \) is (Fréchet) differentiable in \( x \in U \) if there exists \( T_x \in \mathcal{B}(X,Y) \) such that
\[ \lim_{h \to 0} \frac{\|f(x+h) - f(x) - T_x(h)\|_Y}{\|h\|_X} = 0. \]

If such a \( T_x \) exists it is usually denoted by \( Df|_x \) or \( df(x) \) and called the (Fréchet) derivative of \( f \) in \( x \in U \). The map \( f \) is (Fréchet) differentiable if \( f \) is (Fréchet) differentiable in all \( x \in U \).

Higher regularity is defined as follows.

**Definition 4.21.** Let \( X, Y \) be Banach spaces, \( U \subseteq X \) open and \( f : U \to Y \) Fréchet differentiable. \( f \) is \( C^k \) for \( k \in \mathbb{N}^+ \) if the following map is \( C^{k-1} \) :
\[ U \to \mathcal{B}(X,Y), \quad x \mapsto Df|_x \]
There is also a weaker notion of differentiability that resembles the ‘directional derivative in all directions’ \( h \in X \):

**Definition 4.22.** Let \( X,Y \) be Banach spaces over \( \mathbb{K} \in \{ \mathbb{R}, \mathbb{C} \} \) and \( U \subseteq X \) open. \( f : U \to Y \) is **Gâteaux differentiable** in \( x \in U \) if the limit

\[
\lim_{t \to 0} \frac{1}{t}(f(x + th) - f(x))
\]

exists for all \( h \in X \).

We note the following.

**Remark 4.23.** Fréchet differentiable \( \Rightarrow \) Gateaux differentiable.

The **Inverse function theorem** generalizes verbatim to Banach spaces:

**Theorem 4.24 (Inverse function theorem).** Let \( X,Y \) be Banach spaces and \( U \subseteq X \) open, and \( f : U \to Y \) a \( C^k \)-map with \( k \in \mathbb{N} \geq 1 \). Let \( x_0 \in U \) with \( Df|_{x_0} \in \mathcal{B}(X,Y) \) bijective. Then there exists an open neighbourhood \( U_0 \subseteq U \) of \( x_0 \) such that the restriction \( f|_{U_0} : U_0 \to Y \) is injective, \( V_0 := f(U_0) \) is open in \( Y \) and

\[
(f|_{U_0})^{-1} : V_0 \to U_0 \text{ is } C^k \quad \text{and} \quad (Df^{-1})|_y = (Df|_{f^{-1}(y)})^{-1} \quad \forall \ y \in V_0.
\]

**Proof.** Cf. appendix of [McDuff & Salamon]. □

When generalizing the **Implicit function theorem**, things get more interesting. First we define

**Definition 4.25.** Let \( X,Y \) be Banach spaces, \( U \subseteq X \) open and pathconnected, and \( f : U \to Y \) a \( C^k \)-map.

1) \( f \) is a **Fredholm map** if \( Df|_x \) is a Fredholm operator for all \( x \in U \).

2) If \( f \) is a Fredholm map we define its **Fredholm index** via

\[
\text{Ind}(f) := \text{Ind}(Df|_x) \quad \forall \ x \in U
\]

which is welldefined due to Corollary 4.19.

Moreover, we need

**Definition 4.26.** Let \( X,Y \) be Banach spaces, \( U \subseteq X \) open and \( f : U \to Y \) a \( C^k \)-map. \( y \in Y \) is a **regular value** of \( f \) if \( Df|_x \in \mathcal{B}(X,Y) \) is surjective for all \( x \in f^{-1}(y) \subseteq U \).

The following theorem displays the geometric implications Fredholm maps have.

**Theorem 4.27 (Implicit function theorem).** Let \( X,Y \) be Banach spaces, \( U \subseteq X \) open, \( f : U \to Y \) a \( C^k \)-map with \( k \in \mathbb{N} \geq 1 \), and \( y \in Y \) a regular value
of $f$. Then $M := f^{-1}(y) \subseteq X$ is a $C^k$-Banach manifold whose tangent space satisfies

$$T_xM = \ker Df|_x \quad \forall \ x \in M.$$ 

If $f$ is Fredholm then $M$ is a finite dimensional manifold where the dimension of the connected component of $M$ containing $x$ is given by

$$\dim(T_xM) = \dim(\ker(Df|_x)) < \infty.$$ 

**Proof.** Cf. appendix of [McDuff & Salamon].

Theorem 4.27 (Implicite function) is often used to show that solution spaces of partial differential equations (PDE) are smooth manifolds. The idea is to consider the PDE $F(u) = 0$ as map $F$ between suitable Banach spaces and show that it is in fact a Fredholm map. Then the solution space

$$\{u | F(u) = 0\} = F^{-1}(0)$$

is a smooth manifold of dimension $\text{Ind}(F)$ if $0$ is a regular value for $F$.

It remains to inquire how ‘typical’ it is for a value $y \in Y$ to be regular.

**Theorem 4.28 (Sard’s Theorem).** Let $X, Y$ be separable Banach spaces, $U \subseteq X$ open, $f : U \to Y$ a $C^k$-map with $k \geq \max\{1, \text{Ind}(f) + 1\}$. Then

$$Y_{\text{reg}}(f) := \{y \in Y | y \text{ regular value of } f\}$$

is of second Baire category, i.e., it is a countable intersection of open and dense sets.

**Proof.** Cf. appendix of [McDuff & Salamon].

This means that ‘most’ values of Fredholm maps are regular.
CHAPTER 5

Sobolev spaces and regularity theory

A very good reference for Sobolev theory is [Adams & Fournier] or the first edition of that book with Adams as single author. The embedding theorems presented in this chapter can also be found in [Alt], but the proofs there are often quite cumbersome.

Our conventions are $\mathbb{N} := \{1, 2, 3, \ldots \}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and, throughout this chapter, we assume $m, m_1, m_2, k \in \mathbb{N}_0$ and $p, p_1, p_2 \in \mathbb{N} \cup \{\infty\}$ such that expressions like $1 \leq m$ or $1 < p < \infty$ stand for $m \in \{1, 2, 3, \ldots \}$ and $p \in \{2, 3, 4, \ldots \}$.

1. Definition and basic properties of Sobolev spaces

We recall some notions from measure theory. Let $m \in \mathbb{N}_0$ and $p \in \mathbb{N} \cup \{\infty\}$ and $\Omega \subseteq \mathbb{R}^n$ open and let $\lambda$ be the Lebesgue measure on $\Omega$. The Lebesgue space of regularity $p$ is given by

$$L^p(\Omega) := \{ f : \Omega \to \mathbb{R} \mid f \text{ is } \lambda\text{-measurable, } \|f\|_{L^p(\Omega)} < \infty \}/\sim$$

where $f \sim g \iff f = g \text{ } \lambda\text{-almost everywhere}$

and, for $p < \infty$,

$$\|f\|_{L^p(\Omega)} := \left( \int_{\Omega} |f|^p \, d\lambda \right)^{\frac{1}{p}}$$

and, for $p = \infty$,

$$\|f\|_{L^\infty(\Omega)} := \sup_{x \in \Omega \setminus N} |f(x)|.$$ 

The elements of a Lebesgue space are called Lebesgue functions.

Remark 5.1. $(L^p(\Omega), \| \cdot \|_{L^p(\Omega)})$ is a Banach space.
A multi index of length \( k \in \mathbb{N}_0 \) on \( \mathbb{R}^n \) is given by a tuple
\[
s = (s_1, \ldots, s_k) \in \{0, 1, \ldots, n\}^k
\]
and we denote by \(|s| := k\) the length of \( s \). Given a smooth function \( f \), we abbreviate de higher partial derivatives via
\[
\partial_s f := \partial_{s_1} \partial_{s_2} \ldots \partial_{s_k} f
\]
and define for \( f \in C^\infty(\Omega, \mathbb{R}) \) the norm
\[
\|f\|_{m,p} := \sum_{|s| \leq m} \|\partial_s f\|_{L^p(\Omega)}.
\]
The space
\[
X^{m,p} := X^{m,p}(\Omega) := \{ f \in C^\infty(\Omega, \mathbb{R}) \mid \|f\|_{m,p} < \infty \},
\]
endowed with the norm \( \| \cdot \|_{m,p} \) is not complete. To obtain a Banach space, we pass to its completion
\[
\hat{X}^{m,p} := \hat{X}^{m,p}(\Omega) := \{ (f_j)_{j \in \mathbb{N}} \mid (f_j)_{j \in \mathbb{N}} \text{ Cauchy sequence in } X^{m,p}(\Omega) \} / \sim
\]
where
\[
(f_j)_{j \in \mathbb{N}} \sim (g_j)_{j \in \mathbb{N}} \iff \lim_{j \to \infty} \|f_j - g_j\|_{m,p} = 0.
\]
We endow it with the metric
\[
\|(f_j)_{j \in \mathbb{N}} - (g_j)_{j \in \mathbb{N})\|_{\hat{X}^{m,p}} := \lim_{j \to \infty} \|f_j - g_j\|_{m,p}
\]
resp. norm
\[
\|(f_j)_{j \in \mathbb{N}}\|_{\hat{X}^{m,p}} := \lim_{j \to \infty} \|f_j\|_{m,p}.
\]

**Remark 5.2.**
1) \( (\hat{X}^{m,p}, \| \cdot \|_{\hat{X}^{m,p}}) \) is complete, thus Banach.
2) \( J : X^{m,p} \to \hat{X}^{m,p}, J(f) := (f, f, f, f, \ldots) \) is isometric and injective.
3) \( X^{m,p} \) is dense in \( \hat{X}^{m,p} \) since \( (f_j)_{j \in \mathbb{N}} \in \hat{X}^{m,p} \) is for example approximated by the sequence \( J(f_1), J(f_2), J(f_3), \ldots \).

The transition from \( X^{m,p} \) to \( \hat{X}^{m,p} \) yields a Banach space, but its topological definition is cumbersome to work with for geometry and analysis purposes. Let us look for a more analysis/geometry friendly description.

Before we give the actual definition, let us gain some intuition. \( (f_j)_{j \in \mathbb{N}} \) being an element of \( \hat{X}^{m,p} \) means that \( (f_j)_{j \in \mathbb{N}} \) is a Cauchy sequence in \( X^{m,p} \), i.e.,
\[
\forall \varepsilon > 0 \exists N \in \mathbb{N} : \forall j, k \geq N : \|f_j - f_k\|_{m,p} < \varepsilon
\]
Because of \( \|f_j - f_k\|_{m,p} = \sum_{|s| \leq m} \|\partial_s f_j - \partial_s f_k\|_{L^p(\Omega)} \), the sequences \( (\partial_s f_j)_{j \in \mathbb{N}} \) are Cauchy sequences in the Banach spaces \( (L^p(\Omega), \| \cdot \|_{L^p(\Omega)}) \) for all \( |s| \leq m \).
Thus $\lim_{j \to \infty} \partial_s f_j =: f^{(s)} \in L^p(\Omega)$ exists w.r.t. convergence in $\| \cdot \|_{L^p(\Omega)}$ and in particular $\lim_{j \to \infty} f_j =: f$. For all $|s| \leq m$ and for all $\varphi \in C_c^\infty(\Omega, \mathbb{R}) := \{ \varphi \in C^\infty(\Omega, \mathbb{R}) | \varphi \text{ has compact support} \}$, we obtain by partial integration
\[
\int_{\Omega} f_j \partial_s \varphi \, d\lambda = (-1)^{|s|} \int_{\Omega} (\partial_s f_j) \varphi \, d\lambda
\]
Using the theorem of dominated convergence, we can pass to the $L^p$-limits and obtain the identity
\[(5.3) \quad \int_{\Omega} f \partial_s \varphi \, d\lambda = (-1)^{|s|} \int_{\Omega} f^{(s)} \varphi \, d\lambda.\]

If $f$ is smooth then $f^{(s)}$ coincides with the derivative $\partial_s f$. If $f$ is not differentiable in the classical sense we set $\partial_s f := f^{(s)}$ and speak of a weak derivative. This motivates the definition of the space
\[
H^{m,p}(\Omega) := \left\{ f \in L^p(\Omega) \left| \begin{array}{l}
\forall |s| \leq m : \exists f^{(s)} \in L^p(\Omega) : f^{(0)} = f \\
\int_{\Omega} f \partial_s \varphi \, d\lambda = (-1)^{|s|} \int_{\Omega} f^{(s)} \varphi \, d\lambda \quad \forall \varphi \in C_c^\infty(\Omega, \mathbb{R})
\end{array} \right. \right\}
\]
which is often also denoted by $W^{m,p}$ and called Sobolev space of regularity $(m, p)$. Its elements are called Sobolev functions of regularity $(m, p)$. We define
\[
H^{0,p}(\Omega) := L^p(\Omega).
\]
We endow $H^{m,p}(\Omega)$ with the norm
\[
\| f \|_{H^{m,p}(\Omega)} := \sum_{|s| \leq m} \| f^{(s)} \|_{L^p(\Omega)}.
\]
and we call
\[
m - \frac{n}{p}
\]
the Sobolev number of the Sobolev space $H^{m,p}(\Omega)$. It is motivated by the integrability properties of the function $x \mapsto \| x \|^p$ with $\rho \in \mathbb{R}$ around the origin, cf. for example [Bressan, Example 8.23] or [Alt, Section 10.7].

**Lemma 5.4.** $C^\infty(\Omega) \cap H^{m,p}(\Omega)$ is dense in $H^{m,p}(\Omega)$ for $1 \leq p < \infty$ and $0 \leq m$. For $p = \infty$, this is not true.

**Proof.** This is proven using convolution with smooth functions with compact support. The details are left to the reader or see [Alt, Theorem 3.28].

Now we come to the relation between $\hat{X}^{m,p}(\Omega)$ and $H^{m,p}(\Omega)$. 


PROPPOSITION 5.5. For all $1 \leq p \leq \infty$ and all $1 \leq m$, \[
\hat{J} : \dot{X}^{m,p}(\Omega) \to H^{m,p}(\Omega), \quad \hat{J}(f_j)_{j \in \mathbb{N}} := \lim_{j \to \infty} f_j \ (\text{limit taken in } L^p(\Omega))
\] is isometric and injective. If $1 \leq p < \infty$ and $1 \leq m$, then $\hat{J}$ is surjective.

Proof. $\hat{J}$ is injective since, given $(f_j)_{j \in \mathbb{N}}, (g_j)_{j \in \mathbb{N}} \in \dot{X}^{m,p}$ with $\hat{J}(f_j)_{j \in \mathbb{N}} = \hat{J}(g_j)_{j \in \mathbb{N}}$, we get \[
0 = \hat{J}((f_j - g_j)_{j \in \mathbb{N}}) = \lim_{j \to \infty} (f_j - g_j) = f - g \quad \text{as } L^p(\Omega)-\text{limiet.}
\]

$\hat{J}$ is isometric: We consider \[
\| (f_j)_{j \in \mathbb{N}} \|_{\dot{X}^{m,p}(\Omega)} = \lim_{j \to \infty} \| f_j \|_{X^{m,p}(\Omega)} = \lim_{j \to \infty} \sum_{|\alpha| \leq m} \| \partial_\alpha f_j \|_{L^p(\Omega)}
\]
\[
= \sum_{|\alpha| \leq m} \lim_{j \to \infty} \| \partial_\alpha f_j \|_{L^p(\Omega)}
\]

Being a Cauchy sequence in $L^p(\Omega)$, the sequence $(\partial_\alpha f_j)_{j \in \mathbb{N}}$ converges in $L^p(\Omega)$. Since the identity (5.3) determines the functions $f^{(s)}$ uniquely in $L^p(\Omega)$, we conclude \[
\cdots = \sum_{|\alpha| \leq m} \| f^{(s)} \|_{L^p(\Omega)}.
\]

For $p \neq \infty$, we conclude surjectivity of $\hat{J}$ via Lemma 5.4. $\square$

Sobolev spaces are very convenient to work with for the following reason.

PROPPOSITION 5.6. Let $\Omega \subseteq \mathbb{R}^n$ be open and $0 \leq m$ and $1 \leq p \leq \infty$. Then $(H^{m,p}(\Omega), \| \cdot \|_{H^{m,p}(\Omega)})$ is a Banach space.

Proof. If $(f_j)_{j \in \mathbb{N}}$ is a Cauchy sequence in $H^{m,p}(\Omega)$ so are the weak derivatives $(\partial_\alpha f_j)_{j \in \mathbb{N}}$ in $L^p(\Omega)$. Since $L^p(\Omega)$ is Banach, the weak derivatives converge to some $f^{(s)} \in L^p(\Omega)$. In particular, $(f_j)_{j \in \mathbb{N}}$ converges to some $f := f^{(0)} \in L^p(\Omega)$. By definition, $f \in H^{m,p}(\Omega)$ and the $f^{(s)}$ are the weak derivatives of $f$. $\square$

If $p \neq \infty$, Proposition 5.6 can also be deduced from Proposition 5.5. We can also define Sobolev spaces with boundary value zero: For $\Omega \subseteq \mathbb{R}^n$ open, $0 \leq m$ and $1 \leq p < \infty$ (note $p \neq \infty$!) we define \[
\overset{\circ}{H}^{m,p}(\Omega) := \{ f \in H^{m,p}(\Omega) \mid \exists (f_j)_{j \in \mathbb{N}} \in C^m_0(\Omega, \mathbb{R}) : \lim_{j \to \infty} \| f - f_j \|_{H^{m,p}(\Omega)} = 0 \}
\]

and endow it with the norm $\| \cdot \|_{\overset{\circ}{H}^{m,p}(\Omega)}$. Then $\overset{\circ}{H}^{m,p}(\Omega)$ is a closed subspace of $H^{m,p}(\Omega)$ and therefore a Banach space. It is compatible with extension of
functions in the following sense. Let $\Omega, \tilde{\Omega} \subseteq \mathbb{R}^n$ open with $\Omega \subseteq \tilde{\Omega}$. Given a function $f \in \dot{H}^{m,p}(\Omega)$, the function

$$f^\ast := \begin{cases} f & \text{on } \Omega, \\ 0 & \text{on } \tilde{\Omega} \setminus \Omega \end{cases}$$

can be approximated by functions from $C_c^{\infty}(\tilde{\Omega}, \mathbb{R})$ and is thus an element of $\dot{H}^{m,p}(\tilde{\Omega})$.

2. Weak and strong convergence in Sobolev spaces

Strong convergence always implies weak convergence as we saw in Proposition 3.4, but sometimes one can pass from weak back to strong convergence at a certain ‘cost’. This is possible in Sobolev spaces and the ‘price to pay’ is the loss of a derivative, i.e., loss of regularity, as we will see now.

Regularity of Sobolev functions depends to a certain extend on controlling the functions near and on the boundary of the underlying open set $\Omega \subseteq \mathbb{R}^n$. On the one hand, the requirements on the boundary should be sufficiently strong to allow reasonable proofs, but, on the other hand, loose enough to admit the domains of standard PDE problems which are often of rectangular shape. Thus requiring Lipschitz regularity for the boundary is a reasonable choice.

**Definition 5.7.** Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. $\Omega$ is said to have **Lipschitz boundary** if its boundary $\partial \Omega$ can be covered by a finite number of open sets $U_1, \ldots, U_k$ such that we have for all $1 \leq j \leq k$:

- $\Omega$ lies on precisely one side of $\partial \Omega \cap U_j$ in $\Omega \cap U_j$.
- $\partial \Omega \cap U_j$ can (maybe after a rotation) be seen as the graph of a Lipschitz function.

We note

**Remark 5.8.** Weak convergence of a sequence $(u_k)_{k \in \mathbb{N}}$ in $H^{m,p}(\Omega)$ is equivalent to weak convergence of the (weak) derivatives $\partial_s u_k$ in $L^p(\Omega)$ for all $|s| \leq m$.

The following result is named after the Austrian-German mathematician Franz Rellich (1906 – 1955).

**Theorem 5.9 (Rellich).** Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with Lipschitz boundary. Let $1 \leq p < \infty$ and $1 \leq m$ and $(u_k)_{k \in \mathbb{N}} \in H^{m,p}(\Omega)$ and $u \in H^{m-1,p}(\Omega)$. Then

1) If $(u_k)_{k \in \mathbb{N}} \in H^{m,p}(\Omega)$ is bounded and $u_k \rightharpoonup u$ converges weakly in $H^{m,p}(\Omega)$ then $u_k \to u$ converges strongly in $H^{m-1,p}(\Omega)$. 
2) Statement 1) also holds true with \( H^{m,p}(\Omega) \) and \( H^{m-1,p}(\Omega) \) replaced by \( \overset{\circ}{H}^{m,p}(\Omega) \) and \( \overset{\circ}{H}^{m-1,p}(\Omega) \). The hypothesis ‘Lipschitz boundary’ is in this situation not necessary.

Proof. See for example [Alt, Sections A 8.1 and A 8.4]. □

Thus the price we pay for gaining strong convergence from weak convergence is the loss of one (weak) derivative.

Values of Sobolev functions are welldefined up to change on a set of Lebesgue measure zero in the domain of definition. If we want to study Sobolev functions on sets of measure zero – like Lipschitz boundaries – we have to make sure that the function is welldefined on the zero set as element of some function space. If the equivalence class contains continuous or even smooth representatives there is no problem, but this is not always the case. Geometrically, the following statement describes when ‘weak boundary values’ are welldefined.

**Theorem 5.10 (Trace operator).** Let \( \Omega \subseteq \mathbb{R}^n \) be open and bounded with Lipschitz boundary and let \( 1 \leq p \leq \infty \). Then there exists exactly one operator \( T : H^{1,p}(\Omega) \to L^p(\partial\Omega) \) such that \( T(u) = u|_{\partial\Omega} \) for all \( u \in C^0(\Omega, \mathbb{R}) \cap H^{1,p}(\Omega) \). This operator is called Trace operator.

Proof. See for example [Alt, Section A 8.6]. □

Thus the price we pay for gaining (weak) boundary values is the loss of one (weak) derivative. The name Trace operator has noting to do with the trace of a matrix. It rather describes the ‘trace’ of \( \partial\Omega \) as ‘curve’ in the graph of a Sobolev function.

**Theorem 5.11 (Rellich’s trace theorem).** Let \( \Omega \subseteq \mathbb{R}^n \) be open and bounded with Lipschitz boundary. Let \( 1 \leq p < \infty \) and \( u, (u_k)_{k \in \mathbb{N}} \in H^{m,p}(\Omega) \). If \( u_k \to u \) converges weakly in \( H^{m,p}(\Omega) \) then \( u_k \to u \) converges strongly in \( L^p(\partial\Omega) \).

Proof. See for example [Alt, Section A 8.13]. □

### 3. Hölder spaces

We want to investigate the interaction between Sobolev and Lebesgue functions on the one hand and ‘classical’ properties like continuity, differentiability, and smoothness on the other hand. Before we delve into this quest, we first generalize the notion of ‘Lipschitz’.

Let \( (Y, |\cdot|) \) be a Banach space, \( \Omega \subseteq \mathbb{R}^n \) and \( ||\cdot|| \) a norm on \( \mathbb{R}^n \). Let \( \alpha \in ]0,1] \) and \( f : \Omega \to Y \) a function. We define the **Hölder constant of** \( f \) **on** \( \Omega \) with
4. Regularity and compactness via embedding theorems

Hölder coefficient $\alpha$ as

$$\text{höl}_\alpha(f, \Omega) := \sup \left\{ \frac{|f(x) - f(y)|}{\|x - y\|^\alpha} \mid x, y \in \Omega, x \neq y \right\} \in [0, \infty].$$

For $\alpha = 1$, the Hölder constant is the Lipschitz constant. For open and bounded $\Omega \subseteq \mathbb{R}^n$ and $k \in \mathbb{N}_0$ and $\alpha \in \{0, 1\}$, the Hölder spaces are defined as

$$C^{k,\alpha}(\overline{\Omega}, Y) := \{ f \in C^k(\overline{\Omega}, Y) \mid \text{höl}_\alpha(\partial_s f, \overline{\Omega}) < \infty \text{ for } |s| = k \}.$$

We endow $C^{k,\alpha}(\overline{\Omega}, Y)$ with the norm

$$\|f\|_{C^{k,\alpha}(\overline{\Omega}, Y)} := \sum_{|s| \leq k} \|\partial_s f\|_{\text{sup}} + \sum_{|s| = k} \text{höl}_\alpha(\partial_s f, \overline{\Omega}).$$

The elements of a Hölder space are called Hölder functions. It makes sense to extend the definition of Hölder spaces to $\alpha = 0$ by setting

$$C^{k,0}(\overline{\Omega}, Y) := C^k(\overline{\Omega}, Y).$$

The Hölder number of the Hölder space $C^{k,\alpha}(\overline{\Omega}, \mathbb{R})$ is given by

$$k + \alpha.$$

Remark 5.12. \( C^{k,\alpha}(\overline{\Omega}, Y), \|\|_{C^{k,\alpha}(\overline{\Omega}, Y)} \) is a Banach space.

4. Regularity and compactness via embedding theorems

On the one hand, Sobolev spaces are necessary since solutions of even ‘smooth’ optimization problems are not necessarily differentiable or continuous. On the other hand, it is certainly nicer to have a ‘classical’ (= continuous/ Lipschitz/ differentiable/ smooth) solution than a ‘weak’ solution in the Sobolev sense.

Since Sobolev functions are in essence equivalence classes of functions we would like to know if/when such an equivalence class contains ‘classical’ representatives. In the end, it comes down to comparing the ‘total regularity’ of the spaces which is given by their Sololev numbers and/or Hölder numbers. To this end, we now present a number of theorems from the literature.

Theorem 5.13 (\( C^{k+1} \hookrightarrow \text{Hölder} \)). Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with Lipschitz boundary. Then, for all $k \in \mathbb{N}_0$,

$$\text{Id} : C^{k+1}(\overline{\Omega}, \mathbb{R}) \to C^{k,1}(\overline{\Omega}, \mathbb{R})$$

is welldefined and continuous.

Proof. See for example [Alt, Section 10.5].
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Theorem 5.14 (Hölder ⇔ Sobolev). Let \( \Omega \subseteq \mathbb{R}^n \) be open and bounded with Lipschitz boundary. Then, for all \( k \in \mathbb{N}_0 \),

\[
\text{Id} : C^{k,1}(\Omega, \mathbb{R}) \rightarrow H^{k+1,\infty}(\Omega, \mathbb{R})
\]

is well-defined. Moreover, it is an isomorphism in the following sense: For all \( u \in H^{k+1,\infty}(\Omega, \mathbb{R}) \) there exists exactly one \( \tilde{u} \in C^{k,1}(\Omega, \mathbb{R}) \) such that \( u = \tilde{u} \) Lebesgue almost everywhere, i.e., \( u = \tilde{u} \) in \( H^{k+1,\infty}(\Omega, \mathbb{R}) \).

Proof. See for example [Alt, Section 10.5]. \qed

Moreover

Theorem 5.15 (Hölder ⇔ Hölder). Let \( \Omega \subseteq \mathbb{R}^n \) be open and bounded. Let \( k_1, k_2 \in \mathbb{N}_0 \) and \( \alpha_1, \alpha_2 \in [0, 1] \) with Hölder numbers satisfying

\[
k_1 + \alpha_1 > k_2 + \alpha_2.
\]

If \( k_1 > 0 \), assume in addition \( \Omega \) to have Lipschitz boundary. Then the embedding

\[
\text{Id} : C^{k_1,\alpha_1}(\Omega, \mathbb{R}) \rightarrow C^{k_2,\alpha_2}(\Omega, \mathbb{R})
\]

is a compact operator.

Proof. See for example [Alt, Section 10.6]. \qed

Moreover

Theorem 5.16 (Sobolev ⇔ Sobolev). Let \( \Omega \subseteq \mathbb{R}^n \) be open and bounded with Lipschitz boundary. Let \( 0 \leq m_1, m_2 \) and \( 1 \leq p < \infty \). Then

1) If

\[
m_1 - \frac{n}{p_1} \geq m_2 - \frac{n}{p_2} \quad \text{and} \quad m_1 \geq m_2
\]

then the embedding \( \text{Id} : H^{m_1,p_1}(\Omega) \rightarrow H^{m_2,p_2}(\Omega) \) is well-defined and continuous. In particular, there exists \( C = C(n, \Omega, m_1, m_2, p_1, p_2) > 0 \) such that

\[
\|u\|_{H^{m_2,p_2}(\Omega)} \leq C \|u\|_{H^{m_1,p_1}(\Omega)}.
\]

2) If

\[
m_1 - \frac{n}{p_1} > m_2 - \frac{n}{p_2} \quad \text{and} \quad m_1 > m_2
\]

then the embedding \( \text{Id} : H^{m_1,p_1}(\Omega) \rightarrow H^{m_2,p_2}(\Omega) \) is compact.

3) Statements analogous to items 1) and 2) hold also true for the spaces \( H^{\tilde{m}_1,p_1}(\Omega) \) and \( H^{\tilde{m}_2,p_2}(\Omega) \). Here the assumption ‘Lipschitz boundary’ is not necessary.

Proof. See for example [Alt, Section 10.9]. \qed

Moreover
Theorem 5.17 (Sobolev \(\hookrightarrow\) Hölder). Let \(\Omega \subseteq \mathbb{R}^n\) be open and bounded with Lipschitz boundary. Let \(1 \leq m\) and \(1 \leq p < \infty\) and \(0 \leq k\) and \(\alpha \in [0, 1]\). Then

1) If

\[ m - \frac{n}{p} = k + \alpha \quad \text{and} \quad 0 < \alpha < 1 \]

then the embedding operator \(\text{Id} : H^{m,p}(\Omega) \rightarrow C^{k,\alpha}(\overline{\Omega})\) is welldefined and continuous, i.e., for all \(u \in H^{m,p}(\Omega)\) there exists exactly one \(\tilde{u} = \text{Id}(u) \in C^{k,\alpha}(\overline{\Omega})\) such that \(u = \tilde{u}\) Lebesgue almost everywhere. In particular, there exists \(C = C(n, \Omega, m, p, k, \alpha) > 0\) such that

\[ \|\text{Id}(u)\|_{C^{k,\alpha}(\overline{\Omega})} \leq C \|u\|_{H^{m,p}(\Omega)} . \]

2) If

\[ m - \frac{n}{p} > k + \alpha \]

then the embedding operator \(\text{Id} : H^{m,p}(\Omega) \rightarrow C^{k,\alpha}(\overline{\Omega})\) is compact.

3) Statements analogous to items 1) and 2) hold also true for the space \(\overset{\circ}{H}^{m,p}(\Omega)\). Here the assumption ‘Lipschitz boundary’ is not necessary.

Proof. See for example [Alt, Section 10.13]. \(\square\)

We summarize the geometric meaning of these embedding theorems:

Remark 5.18. 1) If we can embed ‘space one’ into ‘space two’ then each element of ‘space one’ can be seen as element of ‘space two’. If the elements are equivalence classes of functions, this simply means that each equivalence class contains a representatives that lives in fact in the target space of the embedding.

2) As continuous operator, the embedding implies an estimate of the involved norms.

3) Compact embeddings are useful since composing bounded operators with compact embeddings adjusts the domain or target of the bounded operator suitably to ‘turn the bounded operator compact’.
CHAPTER 6

Elliptic partial differential equations

There are three big classes of partial differential equations (PDEs), namely elliptic, parabolic, and hyperbolic PDEs. In this chapter, we will study elliptic boundary value problems. More precisely, we will state the ‘classical’ (= differentiable or smooth) formulation and then turn it into a ‘weak’ version on Sobolev spaces. Subsequently we will describe how to solve ‘weak’ elliptic PDEs with Dirichlet boundary conditions. A typical example for this type of problems is

\[
\begin{align*}
-\Delta u - u &= f \quad \text{in } \Omega \subseteq \mathbb{R}^n, \\
u &= 0 \quad \text{on } \partial\Omega.
\end{align*}
\]

where \(\Delta := \sum_{i=1}^{n} \partial_{x_i}^2\) is the Laplace operator, the archetypal of elliptic 2nd order differential operators.

1. Classical elliptic boundary value problem

We generalize the boundary value problem in (6.1) as follows. Let \(\Omega \subseteq \mathbb{R}^n\) be open and bounded, \(a_{ij}, b_i \in C^1(\Omega, \mathbb{R})\) for \(1 \leq i, j \leq n\) and \(c, f \in C^0(\bar{\Omega}, \mathbb{R})\). We define the differential operator

\[
L : C^2(\Omega, \mathbb{R}) \to C^0(\Omega, \mathbb{R}),
\]

\[
L(u) := -\sum_{i,j=1}^{n} \partial_{x_j}(a_{ij}(x)\partial_{x_i}u) + \sum_{i=1}^{n} \partial_{x_i}(b_i(x)u) + c(x)u
\]

and want to find a solution \(u \in C^2(\Omega, \mathbb{R}) \cap C^0(\bar{\Omega}, \mathbb{R})\) of the boundary value problem

\[
\begin{align*}
L(u) &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega.
\end{align*}
\]

Boundary conditions of the form \(u = g\) on \(\partial\Omega\) with \(g \in C^0(\partial\Omega, \mathbb{R})\) are called Dirichlet boundary conditions and, in case of \(g \equiv 0\), homogeneous Dirichlet boundary conditions. Note that there is a transformation turning a Dirichlet boundary problem with general \(g\) into a homogeneous one, see [Bressan, Remark 9.5].
Definition 6.4. \( L \) from (6.2) is \textit{elliptic} if there exists \( \theta > 0 \) such that

\[
\xi^T a(x) \xi := \sum_{i,j=1}^{n} a_{i,j}(x) \xi_i \xi_j \geq \theta \| \xi \|^2
\]

for all \( x \in \Omega \) and for all \( \xi \in \mathbb{R}^n \), i.e., the matrix \( a := (a_{i,j})_{1 \leq i,j \leq n} \) is strictly positive definite.

In particular, if the matrix \( a \) of an elliptic operator is diagonalizable then the absolute values of the eigenvalues are bounded from below by \( \theta > 0 \). The name ‘elliptic’ comes from the fact that \( \{ \xi \in \mathbb{R}^n \mid \sum_{i,j=1}^{n} a_{i,j}(x) \xi_i \xi_j = \text{const} \} \) describes under the assumption ‘elliptic’ an ellipse.

Example 6.5. Setting \( (a_{i,j})_{1 \leq i,j \leq n} \) to be the unit matrix, \( b_i \equiv 0 \) and \( c(x) \equiv 1 \) in (6.3), we obtain the boundary value problem (6.1). We may choose \( \theta = 1 \) and obtain that the operator \( \Delta u - u \) in (6.1) is elliptic.

Solution methods for the classical elliptic Dirichlet boundary value problem given by (6.3) can be found for instance in [Farlow].

Apart from being interesting on its own account, the Laplace operator is moreover related to several other important operators:

Remark 6.6. 1) Identify \( \mathbb{R}^2 \approx \mathbb{C} \) via \( (x,y) = x + iy \) and define the operators \( \partial := \partial_x - i \partial_y \) and \( \bar{\partial} := \partial_x + i \partial_y \). Then

\[
\partial \circ \bar{\partial} = \bar{\partial} \circ \partial = \Delta.
\]

\( \partial \) and \( \bar{\partial} \) are so-called \textbf{Cauchy-Riemann operators}.

2) Operators \( \mathcal{D} \) with the property

\[
\mathcal{D}^2 := \mathcal{D} \circ \mathcal{D} = \Delta
\]

are called \textbf{Dirac operators}. They appear for example in Clifford analysis where they are of the form \( \mathcal{D} = \sum_{j=1}^{n} e_j \partial_{x_j} \) with orthonormal frame \( e_1, \ldots, e_n \) of \( \mathbb{R}^n \).

These relations between the various operators are of importance since the operators \( \partial, \bar{\partial}, \mathcal{D} \) ‘inherit’ certain properties from \( \Delta \) such that their analysis profits from the analysis of \( \Delta \) and vice versa.

2. Weak elliptic boundary value problem

In this paragraph, we investigate how to solve (6.3) in case the functions \( a_{ij}, b_i, c, f \) are not any more continuously differentiable or continuous. To this end, let us reformulate the equation \( L(u) = f \) using multiplication by
test functions and integration w.r.t. the Lebesgue measure $\lambda$ on $\mathbb{R}^n$. In the following, we use the short notation

$$
\int_{\Omega}^{} g := \int_{\Omega}^{} g(x) \, d\lambda(x).
$$

For test functions $\varphi \in C_c(\Omega, \mathbb{R})$, we obtain

$$
\int_{\Omega}^{} f \varphi = \int_{\Omega}^{} L(u) \varphi = \int_{\Omega}^{} \left( -\sum_{i,j=1}^{n} \partial_{x_j}(a_{ij}(x)\partial_{x_i} u) + \sum_{i=1}^{n} \partial_{x_i}(b_i(x) u) + c(x) u \right) \varphi
$$

and by means of partial integration

$$
(6.7) \quad = \int_{\Omega}^{} \left( \sum_{i,j=1}^{n} a_{ij}(x)\partial_{x_j} u \partial_{x_i} \varphi - \sum_{i=1}^{n} b_i(x) u \partial_{x_i} \varphi + c(x) u \varphi \right).
$$

To assure well-definedness of the integrals and compliance with the boundary condition $u = 0$ on $\partial\Omega$, the following assumptions are necessary and/or useful:

- $a_{ij}, b_i, c \in L^\infty(\Omega)$ for $1 \leq i, j \leq n$. This gives us the freedom to adjust the conditions on $u$ and $\varphi$ to the geometry of the problem.
- $u$ needs at least one weak derivative. Since $u$ and its weak derivative need to be integrable while being multiplied with $\varphi$, asking $L^2$-integrability of $u$ and its derivative makes sense. This suggests to ask for $H^{1,2}$-regularity of $u$. In order to comply with the boundary condition $u = 0$ on $\partial\Omega$, a natural choice is to work with $u \in H^{1,2}(\Omega)$.
- $\varphi$ starts out from being a smooth test function with compact support, but we may relax this assumption: $\varphi$ needs at least one weak derivative and both $\varphi$ and its derivative need to be integrable while being multiplied with another function. This suggests asking for $L^2$-integrability for $\varphi$ and its derivative, i.e., $H^{1,2}$-regularity of $\varphi$. Since the original $\varphi$ had compact support, the choice of $\varphi \in H^{1,2}(\Omega)$ is suggestive. In addition, this makes $u$ and $\varphi$ being elements of the same space which simplifies the analysis considerably, in particular, since $H^{1,2}(\Omega)$ is a Hilbert space.
• $f$ needs to be integrable while being multiplied with another function. This suggests $f \in L^2(\Omega)$. This is compatible with the regularity of $\varphi$ since the (even compact) embedding $H^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ assures well-definedness of the identification $\int_\Omega f \varphi = \langle f, \varphi \rangle_{L^2}$.

These thoughts lead to

**Definition 6.8.** The **weak elliptic Dirichlet boundary value problem** is defined as follows.

**Given:** $\Omega \subseteq \mathbb{R}^n$ open and bounded, $f \in L^2(\Omega)$, $a_{ij}, b_i, c \in L^\infty(\Omega)$ with $a := (a_{ij})_{1 \leq i, j \leq n}$ elliptic, i.e., there exists $\theta > 0$ such that for Lebesgue-almost all $x \in \Omega$:

$$\xi^T a(x) \xi := \sum_{i,j=1}^n a_{i,j}(x)\xi_i \xi_j \geq \theta \|\xi\|^2.$$ 

**Wanted:** $u \in H^{1,2}(\Omega)$ satisfying

$$\int_\Omega \left( \sum_{i,j=1}^n a_{ij}(x)\partial_{x_i}u \partial_{x_j}\varphi - \sum_{i=1}^n b_i(x)u\partial_{x_i}\varphi + c(x)u\varphi \right) = \int_\Omega f \varphi$$

for all $\varphi \in H^{1,2}(\Omega)$.

Such a Sobolev function $u \in H^{1,2}(\Omega)$ is called a **weak solution** of the elliptic Dirichlet boundary problem

\[
\begin{cases} 
    L(u) = f & \text{in } \Omega, \\
    u = 0 & \text{on } \partial \Omega 
\end{cases}
\]

associated to the ‘formal operator’

$$L(u) := -\sum_{i,j=1}^n \partial_{x_j}(a_{ij}(x)\partial_{x_i}u) + \sum_{i=1}^n \partial_{x_i}(b_i(x)u) + c(x)u.$$
3. POSITIVE DEFINITE OPERATORS AND THE THEOREM OF LAX-MILGRAM

Now we want to make use of the Hilbert space structure of $H^{1,2}(\Omega)$ and $L^2(\Omega)$. Recall the definition of the scalar products

$$\langle g, \varphi \rangle_{L^2} = \int_{\Omega} g \varphi \quad \forall \ g, \varphi \in L^2(\Omega),$$

$$\langle g, \varphi \rangle_{H^{1,2}} = \langle g, \varphi \rangle_{L^2} + \sum_{i=1}^{n} \langle \partial_{x_i} g, \partial_{x_i} \varphi \rangle_{L^2} \quad \forall \ g, \varphi \in L^2(\Omega)$$

and identify the left hand side of equation (6.9) with the bilinear form

$$B : H^{1,2}(\Omega) \times H^{1,2}(\Omega) \to \mathbb{R},$$

$$B(u, \varphi) := \int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij}(x) \partial_{x_i} u \partial_{x_j} \varphi - \sum_{i=1}^{n} b_i(x) u \partial_{x_i} \varphi + c(x) u \varphi \right).$$

We conclude

**Corollary 6.10.** $u \in H^{1,2}(\Omega)$ is a weak solution of (6.9) if and only if $u \in H^{1,2}(\Omega)$ solves the equation

$$B(u, \varphi) = \langle f, \varphi \rangle_{L^2} \quad \forall \ \varphi \in H^{1,2}(\Omega).$$

This transforms the weak elliptic Dirichlet boundary problem into the problem of solving the equation $B(u, \varphi) = \langle f, \varphi \rangle_{L^2}$ for $u \in H^{1,2}(\Omega)$, i.e., we are now looking for some inverse operator between Hilbert spaces.

3. Positive definite operators and the theorem of Lax-Milgram

This section provides us with tools for solving equations involving scalar products and bilinear forms in Hilbert spaces.

**Definition 6.11.** Let $(H, \langle \cdot, \cdot \rangle)$ be a real Hilbert space with induced norm $\| \cdot \| := \sqrt{\langle \cdot, \cdot \rangle}$ and $T : H \to H$ a linear operator. $T$ is strictly positive definite with constant $\theta > 0$ if

$$\theta \| h \|^2 \leq \langle T(h), h \rangle \quad \forall \ h \in H.$$

Strictly positive operators are invertible:

**Theorem 6.12 (Inverse of strictly positive definite operators).** Let $(H, \langle \cdot, \cdot \rangle)$ be a real Hilbert space. Let $T \in \mathcal{B}(H, H)$ be strictly positive
definite with constant $\theta > 0$. Then $T$ has an inverse $T^{-1} \in \mathcal{B}(H, H)$ with operator norm $\|T^{-1}\|_{op} \leq \frac{1}{\theta}$. In particular, for all $f \in H$, there exists a unique ‘solution’ $u \in H$ such that $T(u) = f$, namely $u = T^{-1}(f)$.

**Proof.** Step 1: We show that $T$ is injective. Let $\| \| := \sqrt{\langle , \rangle}$ and estimate

$$\theta \|h\|^2 \leq \langle T(h), h \rangle \leq \|T(h)\| \|h\|.$$ 

This yields $\theta \|h\| \leq \|T(h)\|$ which means that $T(h) = 0$ implies $h = 0$, i.e., $T$ is injective.

Step 2: We show that $T$ is surjective. To this end, we first show that $\text{range}(T)$ is closed. Let $(h_k)_{k \in \mathbb{N}}$ be a sequence in $H$ with existing limit $\lim_{k \to \infty} T(h_k)$. In particular, $(T(h_k))_{k \in \mathbb{N}}$ is a Cauchy sequence which, together with the estimate

$$\|h_k - h_j\| \leq \frac{1}{\theta} \|T(h_k) - T(h_j)\|,$$

implies that $(h_k)_{k \in \mathbb{N}}$ is a Cauchy sequence, too. Since $H$ is Hilbert, the limit $h := \lim_{k \to \infty} h_k$ exists. The continuity of $T$ implies now $\lim_{k \to \infty} T(h_k) = T(h) \in \text{range}(T)$. Since $\text{range}(T)$ is closed, we have the orthogonal decomposition $H = \text{range}(T) \oplus (\text{range}(T))^\perp$. For all $v \in (\text{range}(T))^\perp$, the estimate $\theta \|v\|^2 \leq \langle T(v), v \rangle = 0$ implies $v = 0$, hence $H = \text{range}(T)$.

Step 3: We just proved that $T \in \mathcal{B}(H, H)$ is bijective. Theorem 2.29 (Inverse mapping) implies the existence of the inverse $T^{-1} \in \mathcal{B}(H, H)$. Therefore, given $f \in H$, the equation $T(u) = f$ has the unique solution $u = T^{-1}(f)$. Moreover,

$$\theta \|T^{-1}(f)\| \leq \|T(T^{-1}(f))\| = \|f\|$$

implies $\|T^{-1}\|_{op} \leq \frac{1}{\theta}$. \qed

Recall that a bilinear form is a mapping $B : H \times H \to \mathbb{R}$ that is linear in both variables.

**Definition 6.13.** A bilinear form $B : H \times H \to \mathbb{R}$ is **continuous** if there exists $C > 0$ such that $|B(u, v)| \leq C \|u\| \|v\|$ for all $u, v \in H$. A bilinear form $B$ is **strictly positive definite with constant $\theta > 0$** if there exists $\theta > 0$ such that $\theta \|u\|^2 \leq B(u, u)$ for all $u \in H$.

The following important result is named after the Hungarian-American mathematician Peter D. Lax (born in 1926) and the American mathematician Arthur N. Milgram (1912 – 1961).

**Theorem 6.14 (Lax-Milgram).** Let $(H, \langle , \rangle)$ be a real Hilbert space and $B : H \times H \to \mathbb{R}$ a continuous bilinear form that is strictly positive definite with constant $\theta > 0$. Then, for all $f \in H$, there exists a unique $u_f \in H$ such that $B(u_f, \varphi) = \langle f, \varphi \rangle$ for all $\varphi \in H$. Moreover, $\|u_f\| \leq \frac{1}{\theta} \|f\|$.
Proof. For all \( u \in H \), the map \( B(u, \cdot) : H \rightarrow H, \varphi \mapsto B(u, \varphi) \) is a continuous linear functional. By Theorem 4.7 (Riesz), there exists a unique \( u_B \in H \) such that \( B(u, \varphi) = \langle u_B, \varphi \rangle \) for all \( \varphi \in H \).

Step 1: We show that

\[
(6.15) \quad \beta_B : H \to H, \quad \beta_B(u) := u_B
\]

is a strictly positive definite continuous linear operator: For \( r \in \mathbb{R} \), the identity \( \langle r \beta_B(u), \varphi \rangle = r B(u, \varphi) = B(ru, \varphi) \) implies \( r \beta_B(u) = \beta_B(ru) \). Moreover, we find for \( u_1, u_2, \varphi \in H \):

\[
\langle \beta_B(u_1) + \beta_B(u_2), \varphi \rangle = \langle \beta_B(u_1), \varphi \rangle + \langle \beta_B(u_2), \varphi \rangle = B(u_1, \varphi) + B(u_2, \varphi)
\]

which implies continuity of \( \beta_B \). Moreover, the strictly positive definiteness of \( B \) implies \( \theta \|u\|^2 \leq B(u, u) = \langle \beta_B(u), u \rangle \), i.e., \( \beta_B \) is strictly positive definite.

Step 2: By Theorem 6.12, \( \beta_B \) has an inverse \( (\beta_B)^{-1} \in \mathcal{B}(H, H) \) with operator norm \( \|\beta_B^{-1}\|_{op} \leq \frac{1}{\theta} \). Thus the equation \( \beta_B(u) = f \) has the unique solution \( u_f := \beta_B^{-1}(f) \) and we find

\[
\|u_f\| = \|\beta_B^{-1}(f)\| \leq \|\beta_B^{-1}\|_{op} \|f\| \leq \frac{1}{\theta} \|f\|.
\]

\[ \square \]

4. Solving the weak elliptic Dirichlet boundary problem

We will first consider the solvability of the weak elliptic boundary problem in case \( L \) is a strictly positive definite operator. Then we investigate what happens in the general case.

Theorem 6.16. If \( b_i \equiv 0 \) for all \( 1 \leq i \leq n \) and \( c \equiv 0 \), then the weak elliptic Dirichlet boundary value problem has a unique weak solution, i.e., for all \( f \in L^2(\Omega) \), there exists exactly one \( u_f \in H^{1,2}(\Omega) \) with \( B(u_f, \varphi) = \langle f, \varphi \rangle_{L^2} \) for all \( \varphi \in H^{1,2}(\Omega) \) where \( B \) is the associated bilinear form in Corollary 6.10 with \( b_i \equiv 0 \) for all \( 1 \leq i \leq n \) and \( c \equiv 0 \). Moreover, the ‘inverse’ operator

\[
L^{-1} : L^2(\Omega) \rightarrow H^{1,2}(\Omega), \quad L^{-1}(f) := u_f
\]

is compact.

For the proof, we need the following estimate.
Lemma 6.17 (Poincaré inequality). Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. Then there exists $C > 0$ such that

$$\|u\|_{L^2(\Omega)} \leq C \|\partial_x u\|_{L^2(\Omega)} \quad \forall \ u \in H^{1,2}(\Omega) \text{ and } \forall \ 1 \leq i \leq n.$$  

In particular, for $\nabla u := (\partial_{x_1} u, \ldots, \partial_{x_n} u)$ and $|\nabla u|^2 := \sum_{i=1}^n (\partial_{x_i} u)^2$ and $\|\nabla u\|_{L^2(\Omega)} := \left( \int_\Omega |\nabla u|^2 \right)^{\frac{1}{2}}$, we obtain

$$\|u\|_{L^2(\Omega)} \leq C \sqrt{n} \|\nabla u\|_{L^2(\Omega)}.$$

**Proof.** Left as an exercise to the reader. Otherwise see for instance [Alt] or [Bressan].

In fact, Lemma 6.17 (Poincaré inequality) belongs rightfully in Section 5.4 before the Hölder and Sobolev embedding theorems since it is needed for their proofs (which we skipped).

**Proof of Theorem 6.16.** We want to apply Theorem 6.14 (Lax-Milgram), but this needs some preparations.

**Step 1:** The bilinear form

$$(6.18) \quad B : H^{1,2}(\Omega) \times H^{1,2}(\Omega) \to \mathbb{R}, \quad B(u, \varphi) = \int_\Omega \sum_{i,j=1}^n a_{ij}(x) \partial_{x_i} u \partial_{x_j} \varphi$$

is bounded as the following estimates will show: First consider

$$|B(u, \varphi)| \leq \sum_{i,j=1}^n \int_\Omega |a_{ij}(x)\partial_{x_i} u \partial_{x_j} \varphi| \leq \sum_{i,j=1}^n \|a_{ij}\|_{L^\infty(\Omega)} \|\partial_{x_i} u\|_{L^2} \|\partial_{x_j} \varphi\|_{L^2}.$$  

Since $\|a_{ij}\|_{L^\infty(\Omega)} < \infty$ for all $1 \leq i, j \leq n$ and $\|u\|_{H^{1,2}} = \|u\|_{L^2} + \sum_{i=1}^n \|\partial_{x_i} u\|$, there exists $C > 0$ such that we get the estimate

$$\cdots \leq C \|u\|_{H^{1,2}} \|\varphi\|_{H^{1,2}}.$$  

**Step 2:** We show that $B$ is strictly positive definite. Lemma 6.17 (Poincaré inequality) yields the existence of some $\tilde{C} > 0$ such that

$$\|u\|_{L^2} \leq \tilde{C} \|\nabla u\|_{L^2} \quad \forall \ u \in H^{1,2}(\Omega).$$

By assumption, $(a_{ij})_{1 \leq i, j \leq n}$ is elliptic with constant $\theta > 0$ such that we get

$$B(u, u) = \int_\Omega \sum_{i,j=1}^n a_{ij}(x) \partial_{x_i} u \partial_{x_j} u \geq \int_\Omega \sum_{i,j=1}^n \theta \partial_{x_i} u \partial_{x_j} u = \theta \|\nabla u\|_{L^2}^2.$$
Combining both inequalities, we obtain
\[ \|u\|_{H^1} = \|u\|_{L^2} + \| \nabla u \|_{L^2} \leq \tilde{C} \| \nabla u \|_{L^2} + \| \nabla u \|_{L^2} \leq (\tilde{C} + 1) \| \nabla u \|_{L^2} \leq \tilde{C} + 1 \theta B(u,u). \]

Then \( \tilde{\theta} := \frac{\theta}{1 + \tilde{C}} > 0 \) is the constant of strictly positive definiteness of \( B \).

**Step 3:** Since \( B \) is strictly positive definite, Theorem 6.14 (Lax-Milgram) is valid and implies that, for all \( F \in H^{1,2}(\Omega) \), there exists precisely one \( u_F \in H^{1,2}(\Omega) \) such that \( B(u_F, \varphi) = \langle F, \varphi \rangle_{H^{1,2}} \) for all \( \varphi \in H^{1,2}(\Omega) \). More precisely, in analogy to (6.15), we obtain, for the bilinear form in (6.18), the linear operator
\[ \beta_B : H^{1,2}(\Omega) \rightarrow H^{1,2}(\Omega), \quad \beta_B(u) = u_B \]
which is bounded and strictly positive definite with constant \( \tilde{\theta} > 0 \). Its inverse \( \beta_B^{-1} \) gives us \( u_F := \beta_B^{-1}(F) \) satisfying \( B(u_F, \varphi) = \langle F, \varphi \rangle_{H^{1,2}} \).

**Step 4:** By Theorem 5.16 (Sobolev embeddings), the embedding
\[ \text{Id} : H^{1,2}(\Omega) \rightarrow L^2(\Omega) \]
exists and is compact. Since \( L^2(\Omega) \) and \( H^{1,2}(\Omega) \) are Hilbert spaces, we can, according to Theorem 4.7 (Riesz), identify \( (L^2(\Omega))^* \simeq L^2(\Omega) \) and \( (H^{1,2}(\Omega))^* \simeq H^{1,2}(\Omega) \). Thus we can see the adjoint operator
\[ \text{Id}^* : L^2(\Omega) \rightarrow (H^{1,2}(\Omega))^*. \]

Moreover, we get for all \( f \in L^2(\Omega) \) and all \( \varphi \in H^{1,2}(\Omega) \)
\[ \langle f, \text{Id}(\varphi) \rangle_{L^2} = \langle \text{Id}^*(f), \varphi \rangle_{H^{1,2}} = \langle \beta_B(\beta_B^{-1}(\text{Id}^*(f))), \varphi \rangle_{H^{1,2}} = B(\beta_B^{-1}(\text{Id}^*(f)), \varphi). \]

The operator
\[ L^{-1} := \beta_B^{-1} \circ \text{Id}^* : L^2(\Omega) \rightarrow H^{1,2}(\Omega) \]
is compact due to Lemma 4.9 and \( u_f := L^{-1}(f) \) solves \( B(u_f, \varphi) = \langle f, \varphi \rangle_{L^2} \) uniquely for all \( \varphi \in H^{1,2}(\Omega) \). Thus Corollary 6.10 yields the claim. \( \square \)
APPENDIX A

Appendix

1. Basic facts from topology

We recall the following basic definition.

**Definition A.1.** Let $X$ be a set. A **topology** $\mathcal{T}$ on $X$ is a family of subsets of $X$ satisfying the following properties.

1) $X \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$.
2) Let $I$ be an arbitrary index set and $U_i \in \mathcal{T}$ for all $i \in I$. Then we also have $(\bigcup_{i \in I} U_i) \in \mathcal{T}$.
3) Let $I$ be a finite index set and $U_i \in \mathcal{T}$ for all $i \in I$. Then we also have $(\bigcap_{i \in I} U_i) \in \mathcal{T}$.

The tuple $(X, \mathcal{T})$ is said to be a **topological space**. The elements of $\mathcal{T}$ are referred to as **open sets** and their complements in $X$ as **closed sets**.

2. Basic facts concerning ordinary differential equations

We recall the following basic definition.

**Definition A.2.** Let $D \subseteq \mathbb{R}^{n+1}$ be open with variables $(t, x) = (t, x_1, \ldots, x_n)$ and let $f = (f_1, \ldots, f_n) : D \to \mathbb{R}^n$ be a function. A **(system of) ordinary differential equation(s) (ODE)** is an equation of the form

$$x' = f(t, x),$$

i.e., a system

$$\begin{cases} 
    x'_1 = f_1(t, x_1, \ldots, x_n), \\
    \vdots \\
    x'_n = f_n(t, x_1, \ldots, x_n). 
\end{cases}$$

A **solution** of $x' = f(t, x)$ is a curve $x = (x_1, \ldots, x_n) : I \subseteq \mathbb{R} \to \mathbb{R}^n$ defined on an interval $I \subseteq \mathbb{R}$ such that $\{(t, x(t)) \mid t \in I\} \subset D$ and satisfying $x'(t) = f(t, x(t))$ for all $t \in I$.

Existence and uniqueness of solutions are assured by

**Theorem A.3 (Picard-Lindelöf).** If $f$ is continuous and in addition locally lipschitz w.r.t. the $x$-variable, then every initial value problem of $x' = f(t, x)$ has a unique, maximal solution, i.e., for all $(\tau, \xi)$ in the domain of definition,

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there exists precisely one solution $x$ with $x(\tau) = \xi$ and whose interval of
definition is maximal among all possible intervals of definition of solutions
of this initial value problem.

**Proof.** Cf. for example [Hohloch2]. □

**Notation A.4.** The unique solution for the initial value problem $(\tau, \xi)$ is
often written as $x(t) = x(t, \tau, \xi)$ and considered as function of $t$, $\tau$ and $\xi$.

We recall some basic definitions and facts.

**Definition A.5.** 1) An ordinary differential equation $x' = f(t, x)$ is said to
be autonomous if the function $f$ does not depend on $t$. If $f$ depends on
$t$, the equation is called nonautonomous.
2) An ordinary differential equation is complete if all solutions admit $\mathbb{R}$ as
interval of definition.

The solutions of autonomous equations have nice properties:

**Proposition A.6.** Let $M \subseteq \mathbb{R}^n$, $f \in C^k(M, \mathbb{R}^n)$ with $k \geq 1$ and assume
$x' = f(x)$ to be complete with solutions remaining in $M$. Then the map
\[
\Phi : \mathbb{R} \times M \to M, \quad \Phi(t, z) := x(t, 0, z),
\]
called the flow of $x' = f(x)$, is welldefined and satisfies $\Phi \in C^k(\mathbb{R} \times M, M)$. Usually the flow is written as $\Phi_t(x) := \Phi(t, x)$ for all $t \in \mathbb{R}$ and all $x \in M$ and seen as map $\Phi_t : M \to M$. It has the following properties:
1) $\Phi_0(z) = z \quad \forall z \in M$, i.e., $\Phi_0 = \text{Id}$.
2) $\Phi_s(\Phi_t(z)) = \Phi_{s+t}(z) \quad \forall z \in M, \forall s, t \in \mathbb{R}$, briefly $\Phi_{s+t} = \Phi_s \circ \Phi_t$.

**Proof.** Cf. for example [Hohloch2]. □

In particular, $\Phi_t : M \to M$ is invertible with $(\Phi_t)^{-1} = \Phi_{-t}$ for all $t$. Moreover, the flow induces an $\mathbb{R}$-action on $M$ via
\[
\mathbb{R} \times M \to M, \quad (t, z) \mapsto \Phi_t(z).
\]

3. Basic facts concerning differentiable manifolds

**Definition A.7.** Smooth manifold; Riemannian metric; Riemannian mani-
fold, Hessian, gradient

still to be written...

**Definition A.8.** 2-form, closed, nondegenerate

still to be written...
4. Basic facts from measure theory

Theorem A.9. dominated convergence.

still to be written...

5. Hausdorff maximality principle

We recall a few concepts and results from set theory.

Definition A.10. A set $M$ carries a partial order $\leq$ or is referred to as partially ordered by $\leq$ if we have for all $x, y, z \in M$
1) $x \leq x$,
2) $x \leq y$ and $y \leq x$ implies $x = y$,
3) $x \leq y$ and $y \leq z$ implies $x \leq z$.

We are interested in particular in sequences of nested subsets and their behaviour.

Definition A.11. Let $(M, \leq)$ be a partially ordered set and $\tilde{M} \subseteq M$. The subset $\tilde{M}$ is totally ordered if any $x, y \in \tilde{M}$ satisfy either $x \leq y$ or $y \leq x$. A subset $\tilde{M} \subseteq M$ is maximal among all totally ordered sets if $\tilde{M}$ is not contained in any other totally ordered subset of $M$.

The following is a variant of Zorn’s lemma or the axiom of choice.

Theorem A.12 (Hausdorff maximality principle). Let $M$ be a partially ordered set. Then any totally ordered subset of $M$ is contained in a maximal, totally ordered subset of $M$. 
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